Analytical Solution of the Double-Logarithmic Integral Equation with QCD Running Coupling

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Abstract

The analytical solution of the integral equation describing the double-logarithmic contributions to the non-singlet structure function f_{NS} for the running coupling α_s case has been found for any infrared cut-off parameter μ. For $x \to 0$ and $\mu = \Lambda_{QCD}$ the asymptotical formula $f_{NS}(x, Q^2) = g^2 x^{-g}[\ln(Q^2/\mu^2)+\psi(1)-\psi(g)]$ has been obtained where $g = 8/(33-2n_f)$ and $\psi(z)$ denotes the digamma function. Small-x behaviour of f_{NS} for $\mu > \Lambda_{QCD}$ has been studied numerically.

1 Introduction

A theoretical prediction of the small-x behaviour of the non-singlet structure function $f_{NS}(x, Q^2)$ is of great importance for checking of sum rules (Bjorken, Gottfried etc.) containing contributions of very small x which are not experimentally measured. The most popular way to predict the behaviour of $f_{NS}(x, Q^2)$ for $x \ll 1$ in the framework of the perturbative QCD is in extrapolating into this region the Altarelli-Parisi equations which describe DGLAP evolution of the structure functions with respect to $\ln(Q^2)$ [1], [2], [3]. But these equations are valid for the region $\ln(Q^2) \gg \ln(1/x)$ where they sum up all the leading contributions of the Feynman graphs of the ladder type [3]. To sum correctly up all the leading double-logarithmic (DL) contributions in QED at $x \ll 1$ one can apply the integral equation proposed in [4] (see also the review [5]). The generalization of the DL integral equation for QCD when running coupling α_s depends on momenta of virtual particles in the ladder graphs has been done in [6] where properties of numerical solutions have been discussed.

2 General Solution of the Integral Equation

To take into account all the DL contributions to the non-singlet quark structure function $f_{NS}(x, Q^2)$ one is to solve the integral equation [6]

$$A(t, \beta) = \frac{C_F}{2\pi}\alpha_s(t/\beta) + \frac{C_F}{2\pi}\int_\beta^1 \frac{d\beta'}{\beta'}\int_{\mu^2}^{t_m}\frac{dt'}{t'}A(t', \beta')\alpha_s(t_m) \tag{1}$$

and calculate $f_{NS}(x, Q^2)$ making use of the formula

$$f_{NS}(x, Q^2) = \delta(1-x) + \int_{\mu^2}^{S_m} A(t, \tilde{\beta})\frac{dt}{t} \tag{2}$$

with $\tilde{\beta} = x(1 + t/Q^2)$. Here $C_F = (N^2-1)/2N$ for the colour group $SU(N)$, μ denotes the infrared cut-off parameter, $t_m = t\beta'/\beta,\ S_m = Q^2/(4x),\ q^2 = -Q^2 < 0,\ q$ denotes the momentum transferred in lepton-hadron deep inelastic scattering, $x = Q^2/(2pq)$, p is a momentum of a struck quark in a hadron. For the momentum of virtual quark k in the ladder diagram we have used the Sudakov's parametrization $k = -\alpha q + \beta p + k_T,\ t = -k_T^2 > 0$, and analogous representation for k', $t' = -(k'_T)^2$. Applying the well known formula for QCD running coupling $\alpha_s(P^2) = 12\pi/[(33-2n_f)\ln(P^2/\Lambda^2)]$ and using the variables $y = \ln(t/\Lambda^2),\ y' = \ln(t'/\Lambda^2)$, $z = \ln(1/\beta),\ z' = \ln(1/\beta'),\ b = \ln(\mu^2/\Lambda^2)$ we can rewrite equation (1) in the form

$$A(y, z) = \frac{g}{y+z} + \int_0^z dz' \frac{g}{y+z-z'}\int_b^{y+z-z'} dy' A(y', z') \tag{3}$$

where $g = 8/(33-2n_f)$. It is easy to check that the DL integral equation (3) is equivalent to the differential equation

$$y\frac{\partial^2 A}{\partial y^2} - y\frac{\partial^2 A}{\partial y \partial z} + \frac{\partial A}{\partial y} - \frac{\partial A}{\partial z} + gA = 0 \tag{4}$$

with boundary conditions valid when $b \neq 0$

$$A(y, 0) = \frac{g}{y}\ , \tag{5}$$

$$\Big[\frac{\partial A(y,z)}{\partial y} - \frac{\partial A(y,z)}{\partial z}\Big]\Big|_{y=b} = 0\ . \tag{6}$$

Applying the Laplace transformation to equation (4), defining the function $\phi(y,\rho) = \int_0^\infty e^{-\rho z} A(y,z)dz$ and remembering the boundary condition (5) we get

$$y\frac{\partial^2 \phi}{\partial y^2} + (1-\rho y)\frac{\partial \phi}{\partial y} - (\rho - g)\phi = 0 \; . \tag{7}$$

The general solution of (7) looks like ($\lambda = g/\rho$)

$$\phi(y,\rho) = (\rho y)^{-1/2} \exp\{\rho y/2\}[C(\rho)W_{\lambda-1/2,\,0}(\rho y) + S(\rho)W_{1/2-\lambda,\,0}(-\rho y)] \tag{8}$$

where $W_{k,\,m}(x)$ denotes the Whittaker function [7]. From (3) it is easy to get the unequality $|A| \leq (g/b)\exp\{gz\}$ which leads to the relation $|\phi| \leq (g/b)/[Re(\rho) - g]$. The latter unequality is compatible with (8) if $S(\rho) \equiv 0$ as $W_{1/2-\lambda,\,0}(-\rho y)$ grows at large ρy exponentially in accordance with the asymptotical relation [7]

$$W_{k,\,m}(\pm x) = (\pm x)^k e^{\mp x/2}[1 + O(1/x)] \; . \tag{9}$$

The only unknown function in (8) $C(\rho)$ can be found from the boundary condition (6) and is equal to $g\exp\{-\rho b/2\}/W_{\lambda,\,1/2}(\rho b)$. Finally the general solution of (3) at $b \neq 0$ is given by the inverse Laplace transformation

$$A(y,z) = \frac{g}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\rho(\rho y)^{-1/2}\exp\{\rho(z + y/2 - b/2)\}W_{\lambda-1/2,\,0}(\rho y)/W_{\lambda,\,1/2}(\rho b) \; . \tag{10}$$

The arbitrary parameter a in (10) should be chosen so that for $Re\ \rho \geq a$ the integrand would have no singularities.

3 Small-x Behaviour of the Non-Singlet Structure Function

For $\mu = \Lambda$ ($b = 0$) we have the relation $1/W_{\lambda,\,1/2}(0) = \Gamma(1-\lambda)$. Substituting this relation into (10) and making use of the Cauchy theorem for the calculation of the integral in (10) we get the special solution

$$A(y,z) = \frac{g}{y+z} + g^2\int_\theta^\infty \frac{s\ ds}{e^s - 1}\sum_{n=0}^\infty \frac{g^n z^n}{n!(n+1)!}\left[s - \frac{y}{z}\frac{s}{e^s-1}\right]^n \tag{11}$$

where $\theta = \ln(1 + y/z)$. It is easy to prove that at $z \gg 1$ and $y \leq z$ the main contribution in integral (11) gives the term

$$A_0(y,z) = g^2\int_0^\infty e^{-s}\sum_{n=0}^\infty \frac{g^n z^n s^{n+1}}{n!(n+1)!}ds = g^2 e^{gz} \; . \tag{12}$$

Corrections to this simple expression are much smaller than A_0 if the following conditions are valid

$$\exp\{gz/2\} \gg 1, \quad \exp\{gz - 2\sqrt{gy}\} \gg 1. \tag{13}$$

The calculation of $f_{NS}(x,Q^2)$ with using (2) and (12) predicts at $x \ll 1,\ Q^2/\mu^2 \gg 1$ the following behaviour

$$f_{NS}(x,Q^2) = g^2 x^{-g}[\ln(Q^2/\mu^2) - \psi(g) + \psi(1)] \tag{14}$$

where $\psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$ denotes the digamma function [7] and $g = 8/(33 - 2n_f) = 0.32$ for four active flavours ($n_f = 4$) and $g = 0.296$ for $n_f = 3$. For the logarithmic derivative of $f_{NS}(x, Q^2)$ over $\ln(1/x)$ at $\mu = \Lambda$ we have from (14)

$$\gamma = \frac{\partial \ln f_{NS}(x, Q^2)}{\partial \ln(1/x)} = g = \frac{8}{33 - 2n_f} \ . \tag{15}$$

A comparison of the non-singlet structure function calculation with the aid of approximate formula (14) and the precise one making use of formula (10) is presented in Fig. 1. One can see that the difference between the two results is less than 15% for the x and Q^2 ranges $10^{-4} \leq x \leq 10^{-2}$, $Q^2 \leq 300\ (GeV/c)^2$ which are experimentally studied at HERA. Fig. 1 shows that the agreement between formulae (14) and (10) is unexpectedly good even for $x \geq 10^{-4}$ where $gz/2 \leq 1.5$ and the first condition (13) is not valid. The dependence of the logarithmic derivative of f_{NS} over $\ln(1/x)$ on μ, x and Q^2 is shown in Fig. 2. Formulae (10) and (2) have been applied in the calculation. We see that at $\mu = \Lambda$ γ does not practically depend on x and Q^2 in agreement with formula (15) and is approximately equal to $g = 0.32$ for $n_f = 4$. When $\mu > \Lambda$ γ depends both on x and Q^2. For $x \to 0$ $\gamma \to \gamma_0$ with $\gamma_0 < g$. From Fig. 2 we can see that γ_0 decreases when μ increases and is approximately equal to 0.22 at $\mu = 1$ GeV/c. Increasing μ we exclude the contributions of $k_T \leq \mu$. A comparison of the curves presented in Fig. 2 shows an importance of low k_T (≤ 1 GeV/c) contributions to γ. But it is the region where non-perturbative effects are expected to be important. Hence we should expect that these effects will change a value of γ and it will be different from a value predicted by (15) in the framework of the perturbative QCD. It would be interesting to study the non-perturbative effect contributions to γ even in any model.

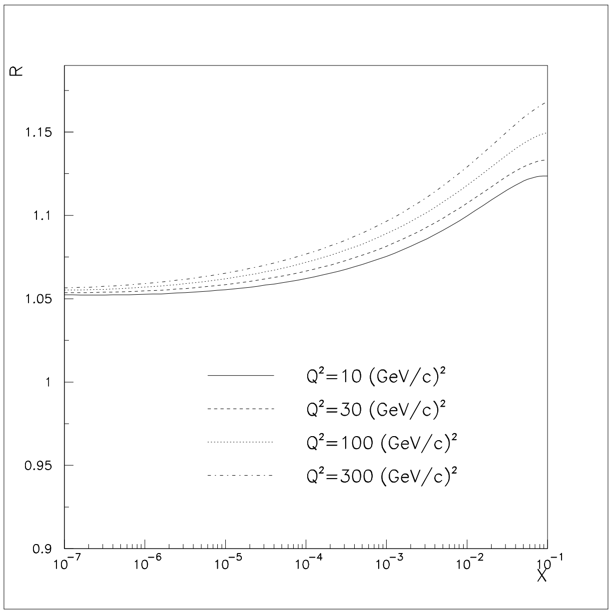

Fig. 1: Ratio of non-singlet structure function given by (14) to that one calculated with the aid of (10), (2) for $\Lambda = 0.2$ GeV/c, $\mu = 0.201$ GeV/c, $n_f = 4$.

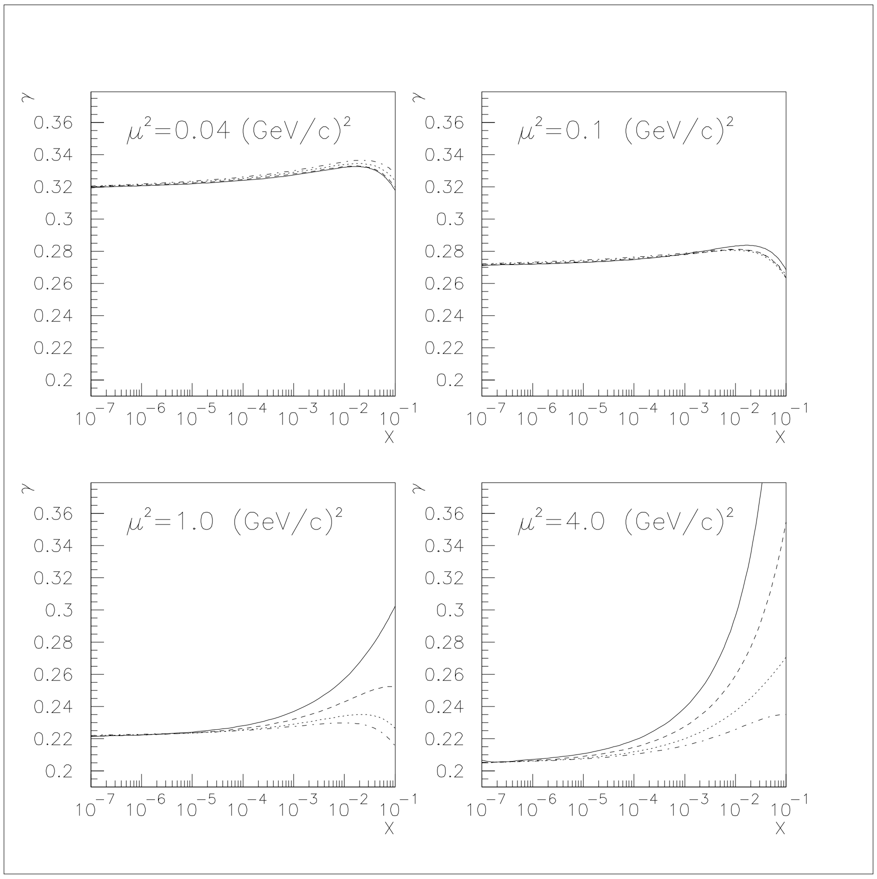

Fig. 2: Dependence of $\gamma = \partial \ln[f_{NS}(x, Q^2)]/\partial \ln(1/x)$ on x, μ and Q^2. Parameters Λ, n_f and Q^2 are the same as in Fig. 1.

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