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New Method for Data Treating in Polarization Measurements

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Abstract

Precise formulas are derived for the expected values $< \xi >$, $< \eta >$ and variances $\delta\xi^2$, $\delta\eta^2$ of random variables ξ, η describing the spin asymmetry of some reaction when a background process contribution is negligible and appreciable, respectively. The variances of ξ and η are proved to be finite. This property differs from that of the Caushy distribution which has an infinite variance. It is shown that $< \xi >$ is equal to the physical asymmetry which allows to find the asymmetry from experimental data without studying the detector efficiency. This is the base of the proposed method of data treating. Asymptotic formulas for $< \eta >$ and $\delta\eta^2$ are also derived for a total number of events tending to infinity for a finite value of the background to signal ratio.

1 Introduction

The study of the polarization phenomena is of great importance in modern physics since it gives a valuable information about the spin structure of interaction under investigation. Typical examples of the polarization measurements to be discussed in the present paper are the measurements of spin-spin asymmetries in inclusive deep inelastic scattering of leptons on nucleons (which give the spin-dependent structure functions of nucleons) or spin-spin asymmetries in semi-inclusive processes. A study of a decaying particle polarization in a final state of some reaction can be performed by measuring spin asymmetries in an angular distribution of detected particles. To get a high statistical accuracy of measured polarization observables we should integrate the result over all kinematic variables of which the studied physical quantity is independent. To perform such an integration we have to know the detector efficiency. Modern detectors are very complicated devices and a problem to describe an efficiency of the particle registration and the detector acceptance is very non-trivial. Therefore approaches in which one does not need the study of the detector efficiency have a great advantage.

Let us consider some examples which explain our approach. To study the polarization of the Λ^0 hyperon we may investigate the angular distribution of the pions and protons in the weak decay

$$\Lambda^0 \to p + \pi^- \ . \tag{1}$$

It is well known that the proton angular distribution in the hyperon rest system looks like

$$W(\vec{S},\vec{P}) = \frac{1}{4\pi}(1+\alpha S\cos\theta) = \frac{1}{4\pi}[1+\alpha(\vec{S}\vec{P})/P] \tag{2}$$

where θ is the angle between the proton three-momentum $\vec{P}$ ($P = |\vec{P}|$) and the hyperon polarization vector $\vec{S}$ ($S = |\vec{S}|$), $\alpha = 0.642 \pm 0.013$ [1] is the well known constant of the weak decay (1). The mean value of the detected number of the protons is

$$\Delta n(\vec{K},\vec{P},\vec{S}) = N_\Lambda E(\vec{K},\vec{P},\vec{R}) W(\vec{S},\vec{P})\Delta\Omega_P \Delta\Omega_K \tag{3}$$

where N_Λ is the total number of the decaying hyperons, $\Delta\Omega_P$ is the solid angle of the registered protons and $\Delta\Omega_K$ denotes the solid angle of the decaying hyperon momenta, $E(\vec{K},\vec{P},\vec{R})$ is the detector efficiency. The latter depends on the hyperon momentum $\vec{K}$, a position of the Λ^0 decay vertex $\vec{R}$ and the proton momentum $\vec{P}$. We would like to remark that we shall not discuss in the present work the problem of smearing which can, in principle, be important for the detector efficiency. We see from (3) that we have to know $E(\vec{K},\vec{P},\vec{R})$, depending on many variables, to obtain the angular distribution $W(\vec{S},\vec{P})$ from the measured quantity $\Delta n/(\Delta\Omega_P\Delta\Omega_K)$. To reduce the statistical uncertainty of the obtained polarization of Λ^0 we are to fit the angular distribution $W(\vec{S},\vec{P})$ within the total detector acceptance for the proton registration.

If it is possible to inverse the direction of the Λ^0 polarization we may study the physical asymmetry

$$C = \frac{\Delta n_+ - \Delta n_-}{\Delta n_+ + \Delta n_-} \tag{4}$$

where

$$\Delta n_\pm = \Delta n(\vec{K},\vec{P},\pm\vec{S})\ . \tag{5}$$

It is easy to see from (3), (2) and (4) that

$$C = \alpha S\cos\theta\ . \tag{6}$$

We see from (6) that C is proportional to the hyperon polarization and does not depend on the detector efficiency if the solid angles $\Delta\Omega_K$ and $\Delta\Omega_P$ are small enough. In this approach we do not need to know the detector efficiency but we have encountered two problems. The quantities of Δn_+ and Δn_- in (3) and (4) are the expected values of the observed numbers p and n of decays (1) for the positive (p) and negative (n) hyperon polarization, respectively. If we consider the random variable ξ

$$\xi = \frac{p-n}{p+n} \tag{7}$$

then its expected value $<\xi>$ can be quite different from the physical asymmetry C given by (4) or by the equivalent relation

$$C = \frac{<p> - <n>}{<p> + <n>} \qquad (8)$$

for finite (and even small) values of $<p>$ and $<n>$. We denote as $<p>$ and $<n>$ in (8) the expected values of the random variables p and n, respectively. We may suspect that $<\xi> \to C$ if $\Delta n_- = <n> \to \infty$, $\Delta n_+ = <p> \to \infty$. The former problem mentioned above is whether the difference $<\xi> - C$ is large or not at finite values of $<p>$ and $<n>$. It will be shown later that $<\xi> = C$ if one may neglect the background contribution.

In the approach under consideration we make use of only a small fraction of experimental events which belong to the solid angle $\Delta\Omega_P$. The latter problem consists in utilization of all the observed events to reduce the statistical uncertainty of the obtained polarization of Λ^0. This problem can be non-trivial if the probability density of ξ has the same property as the Cauchy density. Indeed, the Caushy random variable is equal to the ratio of two random variables having the Gaussian probability densities. It is well known (see for example [2],[3]) that the Caushy distribution has an infinite second moment and the probability to observe values of the Cauchy random variable which deviate significantly from the expected value is large. Moreover, the sample mean of N Cauchy random variables has the same probability density as every random variable. This means that one does not reduce the statistical uncertainty of the measured quantity considering the sample mean. But the random variable ξ is just the ratio of two random variables $p-n$ and $p+n$ (see (7)). We shall show that the variance $\delta\xi^2$ of ξ is finite if we use the conditional probability for ξ and the Poisson densities for the random variables p and n. But it is just the case for the real experimental numbers of events which are positive integers. This result allows to use the total number of events to reduce the statistical uncertainty of the obtained polarization of Λ^0. It is well known how to do this. Indeed, let us divide the total kinematic region of Ω_P into N bins $\Delta\Omega_1$, $\Delta\Omega_2$, ..., $\Delta\Omega_N$ and define the random variables ξ_j, ζ_j in the jth bin by the relations

$$\xi_j = \frac{p_j - n_j}{p_j + n_j} \ , \tag{9}$$

$$\zeta_j = \frac{\xi_j}{b_j} \ , \tag{10}$$

$$b_j = \alpha \cos\theta_j \tag{11}$$

where p_j and n_j denote the numbers of Λ^0 decay events for the positive and negative hyperon polarization, respectively. In (11) θ_j is a value of the angle between the Λ^0 polarization and the proton momentum in jth bin. Let us choose M bins ($M \leq N$) from N bins and denote them as 1th, 2nd, ..., Mth. Since the expected values of ξ_j are equal to C given by (6) (we are to replace θ with θ_j for the jth bin) the random variables ζ_j for every j and ζ have the expected values $< \zeta_j >$ and $< \zeta >$ equal to the Λ^0 polarization S where

$$\zeta = \sum_{j=1}^{M} \beta_j \zeta_j \tag{12}$$

and the coefficients β_j are positive numbers obeying the relation

$$\sum_{j=1}^{M} \beta_j = 1 \ . \tag{13}$$

In a general case b_j are given by expressions other than (11) but they represent some known functions. They are used to make the expected values of all ζ_j equal to each other. It is well known [2] that β_j can be chosen to minimize the variance $\delta\zeta^2$ of the random variable ζ. The optimal choice looks like

$$\beta_j = \frac{1}{\delta\zeta_j^2}\Big[\sum_{m=1}^{M}\frac{1}{\delta\zeta_m^2}\Big]^{-1} \tag{14}$$

and gives the final result for $\delta\zeta^2$

$$\delta\zeta^2 = \Big[\sum_{j=1}^{M}\frac{1}{\delta\zeta_j^2}\Big]^{-1} . \tag{15}$$

Some subtle points concerning properties of ζ will be discussed in the next section.

The method discussed above is applicable also for the case when the detector efficiency is unstable for the time of data taking. The longitudinal double spin asymmetry in inclusive or semi-inclusive deep inelastic scattering of leptons off nucleons is defined by the relation

$$A_{LL} = \frac{d\sigma_{++} - d\sigma_{+-}}{d\sigma_{++} + d\sigma_{+-}} \tag{16}$$

where $d\sigma_{++}$ ($d\sigma_{+-}$) denotes the differential cross section of the studied process when both helicities of colliding particles are positive (have different signs). The typical time of data taking for the modern experiments is about few months. If the detector efficiency and the efficiency of the luminosity monitor changes considerably during this time we cannot use the relations

$$d\sigma_{++} = \frac{n_{++}}{EL_{++}} , \qquad d\sigma_{++} = \frac{n_{+-}}{EL_{+-}} \tag{17}$$

where E denotes as before the detector efficiency, n_{++}, n_{+-} are observed numbers of events and L_{++}, L_{+-} are integrated luminosities when both helicities have the same sign and opposite signs, respectively. If the direction of the target (or beam) polarization changes for example every run we may ignore all the instabilities and define the random variables ζ_j

$$\zeta_j = \frac{n_{++}^{2j-1} - n_{+-}^{2j}}{n_{++}^{2j-1} + n_{+-}^{2j}} \tag{18}$$

for two neighbour runs with numbers $(2j-1)$ and $(2j)$. Formula (18) corresponds to relations (9), (10) with $b_j \equiv 1$. The expected value $< \zeta_j >$ for every ζ_j is equal to the physical asymmetry A_{LL} and hence we may apply formulas (12), (14), (15) to minimize the statistical uncertainty of the measured asymmetry. When writing (18) we supposed that the integrated luminosities are equal to each other for the neighbour runs.

2 Expected value and variance of measured asymmetry

Let us consider the random variable ξ defined by (7) and let the random variables p and n have the Poisson probability densities

$$W_p(p) = \frac{x^p}{p!}e^{-x}\ , \qquad W_n(n) = \frac{y^n}{n!}e^{-y} \tag{19}$$

with the expected values for p and n being equal to x and y, respectively. First we consider the case when the background contribution is negligible. Since x (y) is proportional to the cross section for the case when the product of the initial particle helicities is positive (negative) the physical asymmetry looks like

$$C = \frac{x-y}{x+y} \; . \tag{20}$$

It follows from (7) that ξ becomes meaningless for $p+n=0$, hence we should exclude the term with $p+n=0$ in computing its expected value $<\xi>$. Then the formula for $<\xi>$ reads

$$\begin{aligned} <\xi>= \frac{1}{W_+}\sum_{p+n\geq 1}\frac{x^p}{p!}\frac{y^n}{n!}\frac{p-n}{p+n}e^{-x-y} = \\ \frac{x}{W_+}\sum_{p+n\geq 1}\frac{x^{p-1}}{(p-1)!}\frac{y^n}{n!}\frac{1}{p+n}e^{-x-y} - \\ \frac{y}{W_+}\sum_{p+n\geq 1}\frac{x^p}{p!}\frac{y^{n-1}}{(n-1)!}\frac{1}{p+n}e^{-x-y} \; . \end{aligned} \tag{21}$$

Putting $p+n-1=m$ in the sums in (21) and applying the Newton binomial theorem one gets

$$\begin{aligned} <\xi>= \\ \frac{e^{-x-y}}{W_+}\sum_{m=0}^{\infty}\frac{1}{m+1}\Big[x\sum_{n=0}^{m}\frac{x^{m-n}}{(m-n)!}\frac{y^n}{n!} - y\sum_{p=0}^{m}\frac{x^p}{p!}\frac{y^{m-p}}{(m-p)!}\Big] = \\ (x-y)\frac{e^{-x-y}}{W_+}\sum_{m=0}^{\infty}\frac{(x+y)^m}{(m+1)!} = \Big(\frac{x-y}{x+y}\Big)\frac{1-e^{-x-y}}{W_+} \; . \end{aligned} \tag{22}$$

Since we have excluded in (22) the contribution with $p=n=0$ we are to divide the sum over p and n by the probability of events with $p+n>0$ which is

$$W_+ = 1 - W_p(0)W_n(0) = 1 - e^{-x-y} \; . \tag{23}$$

Combining (22) with (23) we get our final result for the first moment of ξ

$$< \xi >= \frac{x-y}{x+y} \, . \tag{24}$$

Comparison of (24) with (20) shows that the expected value of the random variable ξ coincides with the physical asymmetry C not only for high statistics experiments when $x \gg 1$, $y \gg 1$ but for any values of x and y. The obtained result is rather surprising. It allows to measure the asymmetry in small kinematic regions (bins) and make use of total statistics for all the bins where the expected values of random variables are equal to each other, to improve statistical accuracy of the obtained asymmetry. As has been explained in the Introduction this method allows to avoid a study of the detector efficiency. To realize this program we need the formula for the variance of ξ for every small bin to apply it in formula (15).

To calculate the variance of ξ let us consider the second moment $< \xi^2 >$ and represent it as the sum of four terms I_1, I_2, I_3 and I_4 where

$$< \xi^2 >= \Big\langle \frac{(p-n)^2}{(p+n)^2} \Big\rangle = \Big\langle \frac{p(p-1)}{(p+n)^2} \Big\rangle + \Big\langle \frac{n(n-1)}{(p+n)^2} \Big\rangle - 2\Big\langle \frac{pn}{(p+n)^2} \Big\rangle + \Big\langle \frac{1}{(p+n)} \Big\rangle = I_1 + I_2 + I_3 + I_4 \, . \tag{25}$$

For calculation of I_1 we make use of the relation

$$\frac{1}{(p+n)^q} = \frac{1}{(q-1)!} \int_0^\infty e^{-\alpha(p+n)} \alpha^{q-1} d\alpha \tag{26}$$

for $q = 2$. This gives for I_1

$$I_1 = \Big\langle \frac{p(p-1)}{(p+n)^2} \Big\rangle =$$

$$\frac{1}{W_+}\sum_{p=2}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{x^p}{(p-2)!}\frac{y^n}{n!}e^{-\alpha(p+n)}\alpha d\alpha e^{-x-y} =$$
$$\frac{x^2 e^{-x-y}}{W_+}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\int_0^{\infty}\frac{x^{m-n}}{(m-n)!}\frac{y^n}{n!}e^{-\alpha(m+2)}\alpha d\alpha =$$
$$\frac{x^2}{W_+}e^{-x-y}\int_0^{\infty}\alpha e^{-2\alpha}\exp\{(x+y)e^{-\alpha}\}d\alpha \qquad (27)$$

where W_+ has been defined in (23) and $m = p + n - 2$. After integrating by parts integral (27) can be represented as

$$I_1 = \frac{x^2}{W_+(x+y)}[1 - e^{-x-y} - \phi(x+y)] \qquad (28)$$

where we denote by $\phi(z)$ the function

$$\phi(z) = e^{-z}\int_0^z \frac{e^t - 1}{t}dt = e^{-z}\sum_{m=1}^{\infty}\frac{z^m}{m \cdot m!}\ . \qquad (29)$$

The integrals I_2 and I_3 can be calculated in an analogous way and we get

$$I_1 + I_1 + I_3 = \frac{(x-y)^2}{(x+y)^2}\Big[1 - \frac{\phi(x+y)}{1 - e^{-x-y}}\Big]\ . \qquad (30)$$

We have taken into account (23) in (30). The calculation of I_4 is trivial. Indeed, remembering (23) and introducing $m = p + n$ one has

$$I_4 = \Big\langle \frac{1}{p+n} \Big\rangle =$$
$$\frac{1}{W_+}\sum_{p+n\geq 1}^{\infty}\frac{x^p}{p!}\frac{y^n}{n!}\frac{e^{-x-y}}{p+n} = \frac{e^{-x-y}}{W_+}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=0}^{m}\frac{x^{m-n}}{(m-n)!}\frac{y^n}{n!} =$$
$$\frac{e^{-x-y}}{W_+}\sum_{m=1}^{\infty}\frac{(x+y)^m}{m \cdot m!} = \frac{\phi(x+y)}{1 - e^{-x-y}} \qquad (31)$$

where $\phi(z)$ is defined by (29). Putting (30) and (31) into (25) and remembering (24) we get easily the final formula for the variance of ξ

$$\delta\xi^2 =< \xi^2 > - < \xi >^2 = \frac{\phi(x+y)}{1-e^{-x-y}}\Big[1 - \frac{(x-y)^2}{(x+y)^2}\Big] . \qquad (32)$$

For numerical calculations with formula (32), it is convenient to use the integral representations for $\phi(z)$

$$\phi(z) = -\int_0^z \ln(1 - t/z) e^{-t} dt \ , \qquad (33)$$

$$\phi(z) = 1 - e^{-z} + z^2 \int_0^1 (1-t)\ln(1-t) e^{-zt} dt \ . \qquad (34)$$

Now we are going to discuss some properties of the variance (32). For a high statistics case when $x + y \gg 1$ we may use the asymptotic power series for $\phi(z)$ valid if $z \to \infty$

$$\phi(z) = \sum_{n=0}^{\infty} \frac{n!}{z^{n+1}} \ . \qquad (35)$$

Formula (35) can be easily derived at $z \gg 1$ from relation (33) through decomposition of $\ln(1 - t/z)$ into a power series and integration over t with the limits 0 and ∞. Substituting (35) into (32) we get the asymptotic relation valid at $x + y \gg 1$

$$\delta\xi^2 = \frac{1 - C^2}{x+y} + O((x+y)^{-2}) \qquad (36)$$

where C is the physical asymmetry defined by (20) and $O((x+y)^{-2})$ denotes terms of the order $(x+y)^{-2}$ and smaller at

$x + y \to \infty$. Since x and y are the expected values of observed numbers of events p and n, respectively, formula (36) says that $\delta\xi$ decreases as reciprocal of a square root of the total event number. The solid line in Fig. 1 shows the dependence of $\phi(z)$ on z, and the dash-dotted curve represents a dependence on $z = x + y$ of the ratio of the variance calculated with (32) to its asymptotic expression given by (36). We see that the ratio deviates significantly from the unity at $z \leq 15$ especially at $z \sim 5$. This means that widely used formula (36) for the variance of the measured asymmetry is not valid for the low statistics. We would like to stress that though ξ is the ratio of the random variables $p - n$ and $p + n$ (see(7)) it has a finite statistical uncertainty at any x and y which follows immediately from (32), (34) and (36). This result is in a contrast with the well known property of the Cauchy random variable having the infinite second moment. The reason for this is the used convention of removing the contribution of events with $p + n = 0$. Another difference of principle between the Cauchy distribution and the probability distribution of ξ is as follows. Since for ξ defined by (7) the numerator $p - n$ and the denominator $p + n$ are correlated we have for any integer numbers p, n and k a relation

$$-1 \leq \Big(\frac{p-n}{p+n}\Big)^k \leq 1 \tag{37}$$

which means that all moments of ξ are finite if $p + n > 0$. The Cauchy random variable is a ratio of two uncorrelated random variables. We see also from (32) and (20) that $\delta\xi^2 = 0$ if $|C| = 1$. This property of $\delta\xi^2$ is due to the fact that for $x = 0$ the probability to observe $p > 0$ vanishes according to (19), hence $\xi = -1$ for any n, which means $\delta\xi^2 = 0$. It follows also from (19) that $n \equiv 0$ at $y = 0$, hence $\xi = 1$ for any p. The property under discussion can be proved from the obvious inequality

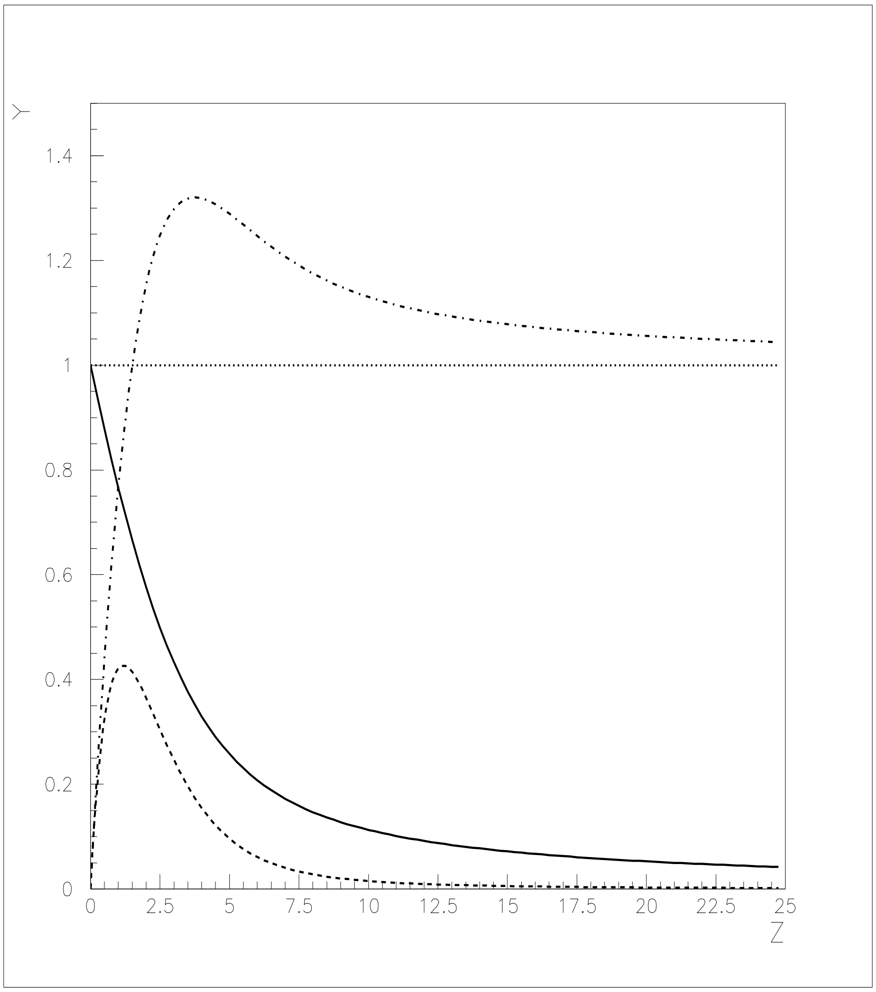

Fig. 1: Functions $\phi(z)$, and $\chi(z)$. Solid and dashed curves show functions $\phi(z)$, $\chi(z)$ defined by (34), (86), respectively. Dash-dotted curve represent the function $z * \phi(z)$.

$$1 \geq <\xi^2> = <[(\xi - <\xi>) + <\xi>]^2> = \delta\xi^2 + <\xi>^2$$

which means that

$$\delta\xi^2 \leq 1 - <\xi>^2 = 1 - C^2 \,. \tag{38}$$

Let us consider the limit of expression (32) at $x + y \to 0$ which will be used below. Applying (34) at low $x + y$ we have

$$\phi(x + y) = x + y + O((x + y)^2) \tag{39}$$

and substituting (39) into (32) we get the relation valid at $x + y \to 0$

$$\delta\xi^2 = 1 - \Big(\frac{x - y}{x + y}\Big)^2 = 1 - <\xi>^2 \tag{40}$$

which realizes the upper limit in inequality (38).

As has been discussed in the Introduction, to increase a statistical accuracy of the measured physical quantity (the polarization, the spin-spin asymmetry etc.) we are to utilize total number of experimental events. For this reason we consider all those bins in which the first moments of ζ_j defined by (9), (10) are equal to each other. But in the practical application of formulas (12), (14), (15) we should remember that ξ_j and ζ_j are not defined for those bins where $p_j + n_j = 0$. Hence we may use M bins among N bins only ($M \leq N$). It is obvious that M cannot be larger than the observed number of the experimental events N^{exp}. It is easy to get the upper limit for the variance of ζ defined by (12) and (14). Let us consider very small bins so that $N \gg N^{exp}$. For this case the expected values of the observed numbers of events p_j and n_j are much less than unity ($x_j \ll 1,\ y_j \ll 1$) and M coincides practically with N^{exp}. We may apply (40) for the variance of ξ_j. Remembering the relation between ξ_j and ζ_j given by (10) we get the following formula for the variance of ζ_j valid if $x_j \ll 1,\ y_j \ll 1$

$$\delta\zeta_j^2 = \frac{1}{b_j^2}(1- <\xi_j>^2) = \frac{1}{b_j^2}(1 - b_j^2 <\zeta_j>^2)\,. \tag{41}$$

Taking into account that all $<\zeta_j>$ are equal to $<\zeta>$ we substitute (41) into (15) and come to the relation

$$\delta\zeta^2 = \Big[\sum_{j=1}^{M} \frac{b_j^2}{1 - b_j^2 <\zeta>^2}\Big]^{-1}\,. \tag{42}$$

Since for physically interesting cases $0 \leq b_j^2 <\zeta>^2 < 1$ we get from (42) the inequality of interest

$$\delta\zeta^2 \leq \frac{1}{M <b^2>_M} \tag{43}$$

where $<b^2>_M$ is the arithmetic mean of b_j^2 over M bins

$$<b^2>_M = \frac{1}{M}\sum_{j=1}^{M} b_j^2\,. \tag{44}$$

Since $M \approx N^{exp}$ formula (43) shows that even for very small bins when observed event numbers p_j, n_j are about 1 we can get high statistical accuracy ($\delta\zeta \sim 1/\sqrt{N^{exp}}$) if $N^{exp} \gg 1$.

The procedure of extraction of the physical quantity $<\zeta>$ and its variance $\delta\zeta^2$ from experimental data can be as follows. First, we calculate the arithmetic mean $\bar{\zeta}$ with the aid of the formula

$$\bar{\zeta} = \frac{1}{M}\sum_{j=1}^{M} \zeta_j \tag{45}$$

where ζ_j has been defined by (9) and (10). The arithmetic mean $\bar{\zeta}$ is an estimate of the physical quantity $< \zeta >$. After that we may estimate parameters x_j and y_j putting

$$x_j = p_j, \quad y_j = n_j \tag{46}$$

and apply (46) to calculate $\delta\bar{\zeta}_j^2$ with the aid of relation (32)

$$\delta\bar{\zeta}_j^2 = \frac{\phi(p_j + n_j)}{b_j^2(1 - e^{-p_j - n_j})}[1 - b_j^2\bar{\zeta}^2] \tag{47}$$

which is an estimate of the variance of ζ_j. Finally, we put $\delta\bar{\zeta}_j^2$ into (15) instead of $\delta\zeta_j^2$ to get an estimate of the variance of ζ where

$$\delta\bar{\zeta}^2 = \Big[\sum_{j=1}^{M} 1/\delta\bar{\zeta}_j^2\Big]^{-1} . \tag{48}$$

Since p and n are integer numbers the random variable ξ is defined for rational numbers in accordance with (7). The probability W_ξ to observe $\xi = (p - n)/(p + n)$ is not a smooth function of ξ. We may formally define the quantity Ω by the relation

$$\Omega(\xi, h) = \frac{1}{h} W_\xi(\xi - \frac{h}{2} < \frac{p - n}{p + n} < \xi + \frac{h}{2}) \tag{49}$$

which would become the probability density in the limit $h \to 0$ if W_ξ were a smooth function . Figure 2 shows the dependence of Ω on ξ at $x = 6$, $y = 2$ for numbers of bins N equal to 30 (top) and $N = 150$ (bottom) where h and N in (49) are related with the formula $h = 2/N$. It is obvious that there is no smooth function Ω given by (49) at $h \to 0$. Figure 3 shows that Ω corresponds to more smooth histograms than those presented in

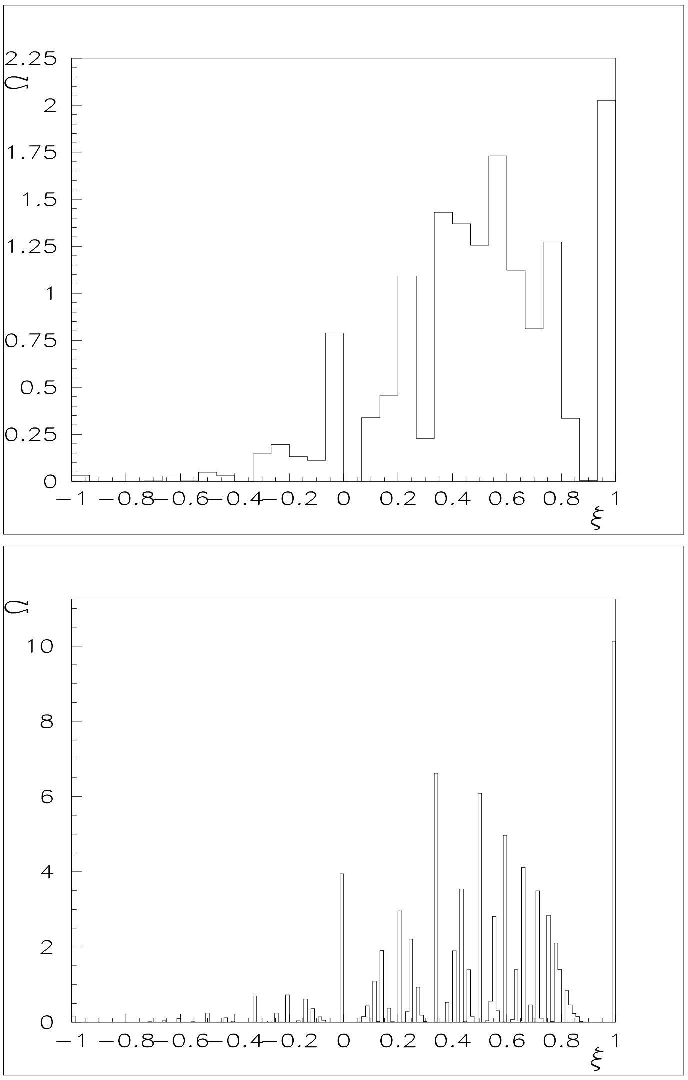

Figure 2: Probability density Ω for random variable ξ. Histograms are computed for $x = 6$ and $y = 2$. Bin number: top - $N = 30$, bottom - $N = 150$.

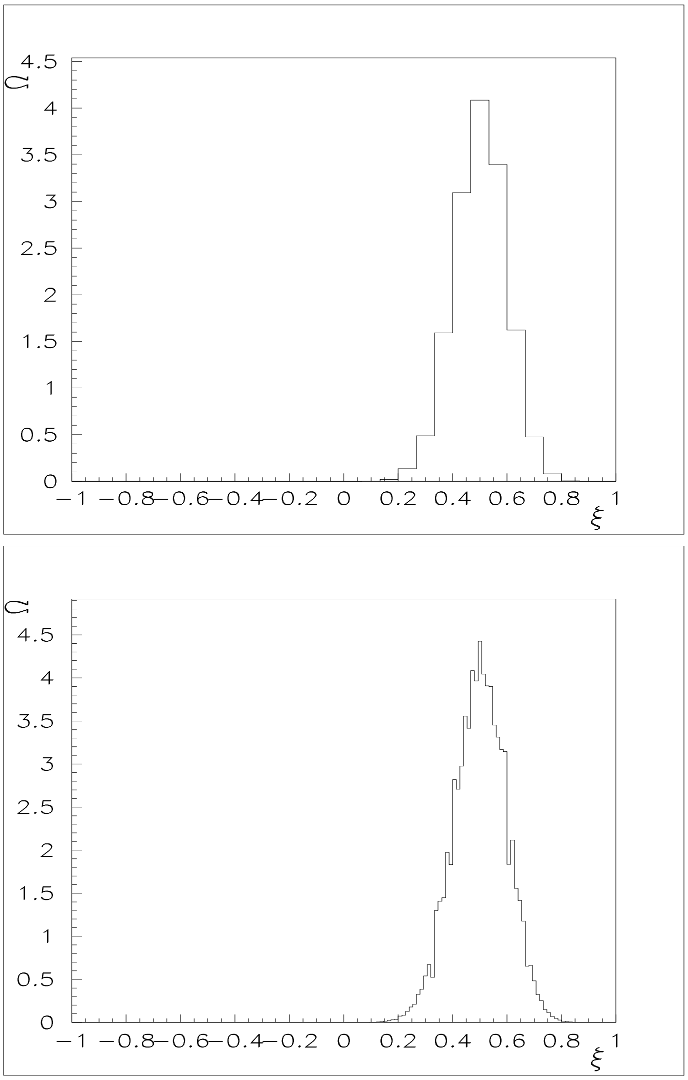

Figure 3: Probability density Ω for random variable ξ. Histograms are computed for $x = 60$ and $y = 20$. Bin number: top - $N = 30$, bottom - $N = 150$

Fig. 2 if we increase x and y keeping the asymmetry C (see (20)) equal to the same value as in Fig. 2. Figure 3 corresponds to higher statistics in comparison with Fig. 2 since $x = 60$, $y = 20$. It is easy to see from Fig. 3 that for high statistics Ω resembles the Gaussian distribution.

To find the confidence levels which correspond to the regions

$$C - k\delta\xi < \xi < C + k\delta\xi \tag{50}$$

($k = 1,\ 2,\ 3$) usually used in the statistical analysis of the experimental data we are to calculate the cumulative distribution using the formula

$$F(z) = W_\xi(\xi < z) = \sum_{\xi < z} \frac{x^p}{p!}\frac{y^n}{n!} e^{-x-y}/W_+ \,. \tag{51}$$

with W_+ given by (23). The sum in (51) runs over all positive integer p and n obeying the unequality $z > \xi = (p-n)/(p+n)$. The values of $< \xi >$, $\delta\xi^2$ are calculated with the aid of (24), (32) and to compute $\omega(k\delta\xi)$ ($k = 1,\ 2,\ ...$)

$$\omega(k\delta\xi) = F(< \xi > +k\delta\xi) - F(< \xi > -k\delta\xi) \tag{52}$$

relation (51) has been used. The results of the calculations are presented in Table 1. The first line in Table 1 corresponds to the Gaussian distribution. We see that even at rather low statistics ($x+y \sim 100$) the numbers $\omega(k\delta\xi)$ agree reasonably with the confidence levels for the normal density function. We see from the comparison of every line with the first line that the higher statistics the better is the agreement between the confidence levels for the random variable ξ with those for the Gaussian distribution. It is easy to see from Table 1 that the agreement between the confidence levels becomes worse with an increase of $| < \xi > |$ (other things being equal) when the expected value goes close to its limits equal to ± 1.

x	y	$< \xi >$	$\delta\xi$	$\omega(\delta\xi)$	$\omega(2\delta\xi)$	$\omega(3\delta\xi)$
				0.68268	0.94601	0.99730
3	1	0.5	0.49712	0.43339	0.95559	0.96794
12	4	0.5	0.2242	0.68386	0.95436	0.99615
120	40	0.5	0.068682	0.68344	0.95427	0.99720
7	1	0.75	0.2535	0.87089	0.96202	0.99453
28	4	0.75	0.11886	0.68360	0.95069	0.99546
140	20	0.75	0.052456	0.68165	0.95492	0.99693
19	1	0.9	0.10016	0.85166	0.95990	0.99207
76	4	0.9	0.049045	0.68116	0.94877	0.99527
380	20	0.9	0.021822	0.68228	0.95547	0.99694

3 Taking into account background events

Usually measured numbers of events consist of events of the process under investigation and background events. Let the background events are described with random variables m, k both having the Poisson distributions

$$W_m(m) = \frac{z^m}{m!}e^{-z} \ , \qquad W_k(k) = \frac{t^k}{k!}e^{-t} \tag{53}$$

where m (k) describes the background contribution to the observed number p (n). Since the expected numbers of the events for the process under investigation are equal to $x - z$ and $y - t$, then the formula for the physical asymmetry reads

$$C = \frac{x - z - y + t}{x + y - z - t} . \tag{54}$$

We suppose as before that the region of the kinematic variables is small enough therefore the detector efficiency can be considered as a constant and hence the ratio in (54) does not depend on the detector efficiency.

We can obtain the asymmetry if

$$p + n - m - k \neq 0 \tag{55}$$

but we start our discussion considering the measured asymmetry in case when a total number of events $p + n$ is larger than a total number of background events $m + k$

$$p + n - m - k > 0 . \tag{56}$$

The total probability to observe events in region (56) will be denoted as W_+ and is given by the relations

$$W_+ = 1 - W_- , \tag{57}$$

$$W_- = \sum_{p+n \leq m+k} \frac{x^p \; y^n \; z^m \; t^k}{p! \; n! \; m! \; k!} e^{-x-y-z-t} =$$

$$\sum_{l=0}^{\infty} \frac{(z+t)^l}{l!} \sum_{q=0}^{l} \frac{(x+y)^q}{q!} \varepsilon = \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \frac{(z+t)^{q+r} (x+y)^q}{(q+r)! \; q!} \varepsilon . \tag{58}$$

For transformation of the sums in (58) we put $q = p+n$, $l = m+k$ and after that we define $r = l - q$. We have applied in (58) the short notation

$$\varepsilon = e^{-x-y-z-t} . \tag{59}$$

Making use of the well known formula [4]

$$\sum_{r=0}^{\infty} \frac{\xi^r}{r!(\nu+r)!} = \xi^{-\nu/2} I_\nu(2\sqrt{\xi}) \tag{60}$$

for the Bessel functions $I_m(z) = (-i)^m J_m(iz)$ depending on the imaginary argument we get from (58)

$$W_- = \sum_{r=0}^{\infty} \Big(\frac{z+t}{x+y}\Big)^{r/2} I_r\Big(2\sqrt{(x+y)(z+t)}\Big) e^{-x-y-z-t} \; . \tag{61}$$

Remembering another well known formula for the Bessel functions [4]

$$J_n(z) = \frac{1}{2\pi i} \oint u^{-n} \exp\{\frac{z}{2}(u+\frac{1}{u})\} \frac{du}{u} \tag{62}$$

and putting

$$u = a\sqrt{\frac{x+y}{z+t}} \exp\{i(\phi - \frac{\pi}{2})\} \tag{63}$$

with arbitrary a obeying the inequality

$$a > 1 \tag{64}$$

we get the useful expression

$$I_n(2\sqrt{(x+y)(z+t)}) = \frac{a^{-n}}{2\pi} \Big(\frac{z+t}{x+y}\Big)^{n/2} \int_0^{2\pi} d\phi e^{-in\phi} E(x+y, z+t, \phi) \tag{65}$$

with

$$E(x+y, z+t, \phi) = \exp\{a(x+y)e^{i\phi} + \frac{1}{a}(z+t)e^{-i\phi}\} \; . \tag{66}$$

Summing over r in (61) with the aid of (65) and remembering (57) we obtain the final result for W_+

$$W_+ = 1 - \frac{\varepsilon}{2\pi} \int_0^{2\pi} d\phi E(x+y, z+t, \phi) \\ \Big[1 - \frac{z+t}{a(x+y)} e^{-i\phi}\Big]^{-1} d\phi \ . \tag{67}$$

Let us define the random variable η by the relation

$$\eta = \frac{(p-m)-(n-k)}{(p-m)+(n-k)} \tag{68}$$

and consider its expected value in region (56)

$$< \eta > W_+ = \varepsilon \sum_{p+n>m+k} \Big(\frac{x^p \ y^n \ z^m \ t^k}{p! \ n! \ m! \ k!}\Big)\Big(\frac{p-n-m+k}{p+n-m-k}\Big) = \\ U_p - U_n - U_m + U_k \ , \tag{69}$$

where for $s = p, \ n, \ m,$ or k

$$U_s = \varepsilon \sum_{p+n>m+k} \Big(\frac{x^p \ y^n \ z^m \ t^k}{p! \ n! \ m! \ k!}\Big)\Big(\frac{s}{p+n-m-k}\Big) \ . \tag{70}$$

Making use of the obvious relation

$$\frac{1}{(p+n-m-k)^j} = \frac{1}{(j-1)!} \int_0^\infty e^{-\alpha(p+n-m-k)} \alpha^{j-1} d\alpha \tag{71}$$

at $j = 1$ (valid in region (56)) we get for U_p

$$U_p = \varepsilon \int_0^\infty d\alpha \sum_{p+n>m+k} \frac{x^p \ y^n \ z^m \ t^k}{(p-1)! \ n! \ m! \ k!} e^{-\alpha(p+n-m-k)} =$$

$$x\varepsilon\int_0^\infty d\alpha e^{-\alpha}\sum_{s+n\geq m+k}\frac{x^s\ y^n\ z^m\ t^k}{s!\ n!\ m!\ k!}e^{-\alpha(s+n-m-k)} =$$
$$x\varepsilon\int_0^\infty d\alpha e^{-\alpha}\sum_{q=0}^\infty\frac{(x+y)^q}{q!}\sum_{l=0}^q\frac{(z+t)^l}{l!}e^{-\alpha(q-l)} =$$
$$x\varepsilon\int_0^\infty d\alpha e^{-\alpha}\sum_{r=0}^\infty e^{-\alpha r}\sum_{l=0}^\infty\frac{(x+y)^{l+r}(z+t)^l}{(l+r)!\ l!}\ . \quad (72)$$

In (72) we put $s=p-1$ and after that introduce $q=s+n$, $l=m+k$. We change the order of summation in (72) making use of $r=q-l$. We get the chain of the equalities substituting (60) and then (65) into (72)

$$U_p = x\varepsilon\int_0^\infty d\alpha e^{-\alpha}\sum_{r=0}^\infty e^{-\alpha r}\Big(\frac{x+y}{z+t}\Big)^{\frac{r}{2}}I_r(2\sqrt{(x+y)(z+t)}) =$$
$$x\varepsilon\int_0^\infty d\alpha e^{-\alpha}\int_0^{2\pi}\frac{d\phi}{2\pi}\sum_{r=0}^\infty e^{-r\alpha}e^{-ir\phi}a^{-r}E(x+y,z+t,\phi) =$$
$$\frac{x\varepsilon}{2\pi}\int_0^{2\pi}d\phi\int_0^\infty d\alpha\Big(\frac{e^{-\alpha}}{1-e^{-\alpha}e^{-i\phi}/a}\Big)E(x+y,z+t,\phi) \quad (73)$$

where $E(x+y,z+t,\phi)$ has been defined by (66). The sum over r in (73) is convergent due to (64). Integrating over α in (73) we get the relation

$$U_p = \frac{ax\varepsilon}{2\pi}\int_0^{2\pi}d\phi e^{i\phi}\ln\Big(\frac{1}{1-e^{-i\phi}/a}\Big)E(x+y,z+t,\phi)\ . \quad (74)$$

The formula for U_n can be obtained from (74) by the change $x\leftrightarrow y$. The analogous calculation gives

$$U_k = \frac{t\varepsilon}{2\pi a}\int_0^{2\pi}d\phi e^{-i\phi}\ln\Big(\frac{1}{1-e^{-i\phi}/a}\Big)E(x+y,z+t,\phi)\ . \quad (75)$$

The relation for U_m follows from (75) if we make use of the transformations $t \leftrightarrow z$. Putting (74), (75) and relations for U_n, U_m into (69) we get the final result for the expected value of the random variable η valid if (56) is fulfilled

$$< \eta >= \frac{\varepsilon}{2\pi W_+} \int_0^{2\pi} d\phi \Big[a(x-y) e^{i\phi} + \frac{t-z}{a} e^{-i\phi} \Big]$$
$$\ln\Big(\frac{1}{1 - e^{-i\phi}/a} \Big) E(x+y, z+t, \phi) \qquad (76)$$

where W_+ is given by (67), $E(x+y, z+t, \phi)$ is defined by (66) and ε is given in (59). We would like to stress that $< \eta >$ does not depend on a if it obeys unequality (64). It is convenient for numerical calculations to put a close to the unity for a better convergency of integrals. To see that $< \eta >$ given by (76) does not coincide with the physical asymmetry C defined in (54) for the case when the background process contribution is important, let us decompose (76) into power series with respect to z and t. Considering them as small parameters and neglecting terms $\sim t^2$, tz, z^2 and terms of higher orders it is easy to get the expression

$$< \eta >= \frac{x-y}{x+y} + 2(tx - yz) \Big[\frac{\phi(x+y) + e^{-x-y}}{1 - e^{-x-y}} - \frac{1}{x+y} \Big] + \ldots \quad . \qquad (77)$$

Expression (77) can be compared with the power series for C

$$C = \frac{x-y}{x+y} + 2\frac{(tx - yz)}{(x+y)^2} + \ldots \qquad (78)$$

which follows from (54). We see from a comparison of (77) and (78) that the difference

$$< \eta > -C = 2(tx - yz) \Big[\frac{\phi(x+y) + e^{-x-y}}{1 - e^{-x-y}} - \frac{1}{x+y} - \frac{1}{(x+y)^2} \Big] + \ldots \qquad (79)$$

is nonzero. Applying (35) we obtain from (79) the asymptotic expression for the difference $<\eta> - C$ valid at $x+y \gg 1$

$$< \eta > -C = \frac{4(tx - yz)}{(x+y)^3} + \dots . \tag{80}$$

Formula (80) becomes especially simple if there is no asymmetry for background events. Indeed, if $z = t$ we get instead of (80)

$$\frac{< \eta > -C}{C} = \frac{2t}{x+y} \cdot \frac{2}{x+y} \dots \quad . \tag{81}$$

Relation (81) shows that the fractional difference between the expected value of the random variable η and the physical asymmetry C is proportional to two small factors: the background to signal ratio, $2t/(x+y)$ and the inverse value of the total number of events, $(x+y)^{-1}$.

To calculate the variance of the random variable η defined by (68) we are to calculate $<\eta^2>$ which (by definition) is given by the formula

$$< \eta^2 > = < \Big(\frac{p-m-n+k}{p+n-m-k}\Big)^2 > =$$

$$= \frac{\varepsilon}{1 - W_-} \sum_{p+n>m+k} \Big(\frac{x^p\ y^n\ z^m\ t^k}{p!\ n!\ m!\ k!}\Big)\Big(\frac{p-n-m+k}{p+n-m-k}\Big)^2 . \tag{82}$$

We make use of formulas (71) with $j = 2$, (60), (65) to calculate sums over p, n, m, k in (82) as it has been demonstrated when obtaining the expected value of η. The final formula for $<\eta^2>$ reads

$$< \eta^2 > = \frac{\varepsilon}{2\pi W_+} \int_0^{2\pi} d\phi \int_0^\infty \frac{d\alpha \alpha e^{-\alpha} E(x+y, z+t, \phi)}{1 - e^{-\alpha} e^{-i\phi}/a}$$

$$\Big[(x-y)^2 e^{-\alpha} + (x+y) + \frac{1}{a^2}(z+t)e^{-2i\phi} +$$
$$\frac{2}{a}(x-y)(t-z)e^{-i\phi} + \frac{1}{a^3}(t-z)^2 e^{-3i\phi}\Big] +$$
$$\frac{\varepsilon(x-y)^2}{W_+}\sqrt{\frac{z+t}{x+y}}I_1\Big(2\sqrt{(x+y)(z+t)}\,\Big) \qquad (83)$$

where W_+, $E(x+y, z+t, \phi)$ and ε are defined in (67), (66) and (59), respectively. To compute the variance of η for region (56) we are to put (83) and (76) into the well known formula

$$\delta\eta^2 = <\eta^2> - <\eta>^2 \quad . \qquad (84)$$

Considering again z and t as small parameters, decomposing $\delta\eta^2$ into power series with respect to z, t and retaining terms $\sim z^0$, z^1, t^0, t^1 only we get for region (56)

$$\delta\eta^2 = \Big[1 - \Big(\frac{x-y}{x+y}\Big)^2\Big]\Big\{\frac{\phi(x+y)}{1-e^{-x-y}} + (z+t)\Big[1 - \frac{\phi(x+y)}{1-e^{-x-y}}$$
$$-(x+y)\phi(x+y)[1-(e^{x+y}-1)^{-2}] - \frac{x+y}{e^{x+y}-1}\Big]\Big\} +$$
$$4\frac{tx+yz}{1-e^{-x-y}}\chi(x+y)$$
$$+4\frac{(tx-yz)(x-y)}{x+y}\Big[\frac{1}{x+y} - \frac{\phi(x+y)+e^{-x-y}}{1-e^{-x-y}}\Big] \quad (85)$$

where $\phi(z)$ is given by (34) and the formula for $\chi(z)$ reads

$$\chi(z) = e^{-z}\int_0^z \frac{e^t}{t}\phi(t)dt = e^{-z}\sum_{m=1}^{\infty}\frac{z^m}{m^2\, m!} =$$
$$\frac{z}{2}\int_0^1 \ln^2(1-t)e^{-zt}dt \ . \qquad (86)$$

The behaviour of the function $\chi(z)$ is presented in Fig. 1 with the dashed curve. To obtain the asymptotic behaviour of $\delta\eta^2$ at large $x+y$, we are to use formula (35) and the asymptotic power series for $\chi(z)$ at $z \to \infty$

$$\chi(z) = \sum_{m=2}^{\infty} \frac{g_m}{z^m} = \frac{1}{z^2} + \frac{3}{z^3} + \frac{11}{z^4} + \frac{50}{z^5} + \dots \qquad (87)$$

where the coefficients g_m in (87) obey the recurrent relation

$$g_m = (m-1)g_{m-1} + (m-2)! \qquad (88)$$

with $g_2 = 1$. Substitution of (35) and (87) into (85) leads to the relation of interest valid at $x+y \to \infty$

$$\delta\eta^2 = \Big[1 - \Big(\frac{x-y}{x+y}\Big)^2\Big]\Big[\frac{1}{x+y} + \frac{1}{(x+y)^2}\Big]$$
$$+4\frac{t+z}{(x+y)^2} - 4\frac{ty^2+zx^2}{(x+y)^4} + \dots \ . \qquad (89)$$

If we consider the expected value of η in region (55) we are to make use of the relation

$$< \eta > (1-W_0) = U_p - U_n - U_m + U_k + V_p - V_n - V_m + V_k \ , \quad (90)$$

where W_0 denotes a total probability of events for which $p+n = m+k$. The formula for W_0 follows immediately from (61) if we keep the term with $r=0$ in the sum and we have as a result

$$W_0 = \varepsilon I_0\Big(2\sqrt{(x+y)(z+t)}\ \Big) = \frac{\varepsilon}{2\pi}\int_0^{2\pi} d\phi E(x+y, z+t, \phi)d\phi \ . \qquad (91)$$

In (90) U_s denote quantities ($s = p,\ n,\ m,\ k$) which have been calculated above and V_s can be defined by relation (70) in which we are to sum over all $p,\ n,\ m,\ k$ in the region

$$p + n < m + k \ . \tag{92}$$

It is easy to check that V_p coincides with $(-U_k)$ in which we are to make substitutions $x \leftrightarrow t,\ y \leftrightarrow z$. The expression for V_k can be obtained from $(-U_p)$ after the same substitutions. Applying the substitutions $x \leftrightarrow y,\ t \leftrightarrow z$ one gets the formulas for V_n and V_m from the relations for V_p and V_k, respectively. Putting expressions for W_0, U_s, V_s ($s = p,\ n,\ m,\ k$) into (90) we get the final formula for the expected value of η in region (55)

$$\begin{aligned} &< \eta >= \\ &\frac{\varepsilon}{2\pi}\int_0^{2\pi} d\phi \Big\{ \Big[a(x-y)e^{i\phi} + \frac{t-z}{a}e^{-i\phi} \Big] E(x+y, z+t, \phi) \\ &\qquad - \Big[a(t-z)e^{i\phi} + \frac{x-y}{a}e^{-i\phi} \Big] E(z+t, x+y, \phi) \Big\} \\ &\qquad \ln\Big(\frac{1}{1 - e^{-i\phi}/a} \Big) \Big[1 - \varepsilon I_0 (2\sqrt{(x+y)(z+t)}) \Big]^{-1} \end{aligned} \tag{93}$$

where ε is defined in (59) and $E(z+t, x+y, \phi)$ is equal to

$$E(z+t, x+y, \phi) = \exp\{ a(t+z)e^{i\phi} + \frac{1}{a}(x+y)e^{-i\phi} \}$$

in accordance with (66).

Decomposing (93) into power series with respect to z and t we get

$$\begin{aligned} &< \eta >= \frac{x-y}{x+y} \\ &+ \frac{2(tx - yz)}{1 - e^{-x-y}} \Big[\phi(x+y) + e^{-x-y} - \frac{1}{x+y} \Big] + \dots \ . \end{aligned} \tag{94}$$

Remembering (78) we get easily for the difference $\eta - C$ the relation

$$< \eta > -C =$$

$$2(tx - yz)\Big[\frac{\phi(x+y) + e^{-x-y} - (x+y)^{-1}}{1 - e^{-x-y}} - \frac{1}{(x+y)^2}\Big] + ... \quad (95)$$

instead of (79). Formula (95) shows that $< \eta >$ does not coincide with the physical asymmetry C in region (55) if the background contribution is not negligible. This is true in spite of applying the subtraction procedure: we consider $p - m$ and $n - k$ instead of p and n in (68). It is easy to check that the first term in the asymptotic formula for $< \eta > -C$ obtained from (95) coincides with (80). If one puts x, y, z, t equal to the observed numbers of the experimental events $x = p$, $y = n$, $z = m$, $t = k$, then formulas (79) and (95) can be used to estimate the systematic uncertainty due to the difference between the expected value of the random variable η and the physical asymmetry C. They can be improved if we make use of the precise expression for C and $< \eta >$ given by (54) and (76), (93), respectively.

To compute the second moment of η in region (55) we may use formula (82) if we add in the sum in the right hand side a contribution of events with $p+n < m+k$ and put W_0 instead of W_-. This contribution is described by formula (83) in which we should use the substitution $x \leftrightarrow t$, $y \leftrightarrow z$, and $W_+ \to 1 - W_0$. The expression for $< \eta^2 >$ looks like

$$< \eta^2 >=$$

$$\frac{\varepsilon}{2\pi(1-W_0)}\int_0^{2\pi} d\phi \int_0^{\infty}\Big(\frac{d\alpha \alpha e^{-\alpha}}{1 - e^{-\alpha}e^{-i\phi}/a}\Big)E(x+y, z+t, \phi)$$

$$\Big[(x-y)^2 e^{-\alpha} + (x+y) + \frac{1}{a^2}(z+t)e^{-2i\phi}$$

$$+\frac{2}{a}(x-y)(t-z)e^{-i\phi}+\frac{1}{a^3}(t-z)^2e^{-3i\phi}\Big]+$$

$$\frac{\varepsilon}{2\pi(1-W_0)}\int_0^{2\pi}d\phi\int_0^{\infty}\Big(\frac{d\alpha\alpha e^{-\alpha}}{1-e^{-\alpha}e^{-i\phi}/a}\Big)E(t+z,x+y,\phi)$$

$$\Big[(t-z)^2e^{-\alpha}+(t+z)+\frac{1}{a^2}(x+y)e^{-2i\phi}$$

$$+\frac{2}{a}(x-y)(t-z)e^{-i\phi}+\frac{1}{a^3}(x-y)^2e^{-3i\phi}\Big]+$$

$$\frac{\varepsilon}{1-W_0}I_1\Big(2\sqrt{(x+y)(z+t)}\,\Big)\Big[(x-y)^2\sqrt{\frac{z+t}{x+y}}$$

$$+(t-z)^2\sqrt{\frac{x+y}{t+z}}\Big] \quad (96)$$

with W_0 defined in (91). Putting (96) and (93) into (84) we come to the formula of interest for the variance of η in the region (55). A decomposition of the formula for the variance into power series up to terms z and t gives the relation

$$\delta\eta^2=\Big[1-\Big(\frac{x-y}{x+y}\Big)^2\Big]\Big\{\frac{\phi(x+y)}{1-e^{-x-y}}+\frac{(z+t)}{1-e^{-x-y}}\Big[1-\frac{\phi(x+y)}{1-e^{-x-y}}$$

$$-(x+y)\phi(x+y)\frac{1-2e^{-x-y}}{1-e^{-x-y}}-(x+y)e^{-x-y}\Big]\Big\}$$

$$+\frac{4(tx+yz)}{1-e^{-x-y}}\chi(x+y)$$

$$+\frac{4(tx-yz)(x-y)}{(x+y)(1-e^{-x-y})}\Big[\frac{1}{x+y}-\phi(x+y)-e^{-x-y}\Big] \quad (97)$$

with $\phi(z)$ and $\chi(z)$ defined in (34) and (86), respectively. The asymptotics of (97) at $x+y\gg 1$ is given by relation (89).

The most important limit of the obtained formulas corresponds to the case when $x+y+z+t$ go to infinity but the ratio

$Q = (z+t)/(x+y)$ is less than the unity. For this case the contribution of the region $p+n < m+k$ is exponentially small. Indeed, making use of the well known asymptotic formula for the Bessel functions [4]

$$I_n(z) = \frac{1}{\sqrt{2\pi z}} e^z \tag{98}$$

valid at $z \to \infty$ we get from (61) the relation

$$\begin{aligned} W_- = \frac{1}{\sqrt{4\pi}} \sum_{m=0}^{\infty} \Big(\frac{z+t}{x+y}\Big)^{m/2} \Big\{ \frac{e^{-x-y-z-t}}{[(x+y)(z+t)]^{1/4}} \Big\} \\ \exp\{2\sqrt{(x+y)(z+t)}\} = \\ \frac{1}{\sqrt{4\pi}} \frac{\sqrt{x+y}\ \exp\{-(\sqrt{x+y}-\sqrt{z+t})^2\}}{(\sqrt{x+y}-\sqrt{t+z})\ [(x+y)(z+t)]^{1/4}} \end{aligned} \tag{99}$$

which shows that $W_- \to 0$ exponentially if

$$(\sqrt{x+y}-\sqrt{z+t})^2 \gg 1 \ . \tag{100}$$

Ignoring such exponentially small corrections we may obtain the asymptotic formula for $<\eta>$ which reads

$$\begin{aligned} <\eta> = \Big(\frac{x-y+t-z}{x+y-t-z}\Big) \Big[1 + \frac{x+y+z+t}{(x+y-z-t)^2} \\ + \frac{3(x+y+z+t)^2}{(x+y-z-t)^4} + \frac{1}{(x+y-z-t)^2} + ...\Big] \\ - \frac{x-y-t+z}{(x+y-t-z)^2} \Big[1 + \frac{3(x+y+z+t)}{(x+y-z-t)^2} + ...\Big] \ . \end{aligned} \tag{101}$$

Comparison of (101) with (54) shows that $<\eta>$ coincides with the physical asymmetry C if

$$\frac{x+y+z+t}{(x+y-z-t)^2} \ll 1 \ . \tag{102}$$

It is easy to see that both conditions (100) and (102) are valid if $x+y+z+t \gg 1$ and $Q=(z+t)/(x+y)<1$. If conditions (100) and (102) are fulfilled the variance of the random variable η looks like

$$\delta\eta^2 = \frac{x+y+z+t}{(x+y-z-t)^2}\Big[1+\frac{(x-y-z+t)^2}{(x+y-z-t)^2}\Big] \\ -2\frac{(x-y)^2-(z-t)^2}{(x+y-z-t)^3}\,. \tag{103}$$

We have retained in (103) the greatest terms in the asymptotic power series only. Formulas (101) and (103) correspond to the Gaussian distribution for the random variable η. In particular, relation (103) can be obtained from the widely used formula

$$\delta\eta^2 = \Big(\frac{\partial\eta}{\partial p}\Big)^2\delta p^2 + \Big(\frac{\partial\eta}{\partial n}\Big)^2\delta n^2 + \Big(\frac{\partial\eta}{\partial m}\Big)^2\delta m^2 + \Big(\frac{\partial\eta}{\partial k}\Big)^2\delta k^2 \tag{104}$$

where the dependence of η on p, n, m, k is given by (68) and $\delta p^2=p$, $\delta n^2=n$, $\delta m^2=m$, $\delta k^2=k$ in accordance with the Poisson distributions (19), (53) for the random variables p, n, m, k. If one can neglect the background contribution ($z\to 0$, $t\to 0$) formula (103) reduces to (36). Formulas (101) and (103) are obtained in Appendix.

Figures 4 and 5 show the dependence of the ratio $f = <\eta>/C$ on λ where C is defined by (54) and the relation between x, y, z, t and λ looks like

$$x=x_0\lambda\,,\quad y=y_0\lambda\,,\quad z=z_0\lambda\,,\quad t=t_0\lambda \tag{105}$$

the parameters x_0, y_0, z_0, t_0 being chosen in such a way that $x_0+y_0+z_0+t_0=1$. The solid lines are calculated in region

(56) with the aid of formula (76) and dashed curves are obtained for region (55) using relation (93). The dash-dotted curves are computed with the asymptotic formula (101). Figure 4 shows that all the ratios go to the unity with an increase of the total statistics but the deviation from the unity increases if the mean value of the background to signal ratio b/s=(z+t)/(x+y-z-t) increases that follows from a comparison of Fig. 4a, Fig. 4b and Fig. 4c for which $b/s = 12.5\%,\ 33\%,\ 75\%$, respectively. We see also that the higher the background to signal ratio the slower a value of $< \eta > /C$ tends to the unity with increasing λ. The deviation from the unity for the curves presented in Fig. 4d is larger than for those of Fig. 4c though the background to signal ratio is the same for these two cases. It is due to the fact that the background asymmetry is equal to zero ($z = t$) for the curves shown in Fig. 4c while it has the opposite sign than the measured asymmetry ($z - t < 0$, $C > 0$) for the case presented in Fig. 4d. The behaviour of the ratio $f =< \eta > /C$ at small C ($C \sim 1\%$) is presented in Fig. 5. Curves in Figs. 5a, 5b, 5c correspond to the physical asymmetry $C = 0.025$ and $b/s = 12.5\%$ but the background events have zero, negative and positive asymmetry, respectively. We see that the greatest deviation of the curves from the unity corresponds to the case when C and the background asymmetry have opposite signs. The curves in Fig. 5d are calculated for the large background contribution ($b/s = 75\%$) nevertheless the maximum deviation of the curves in this case is even smaller than in Figs. 5b, 5c. This example shows that the asymmetry of the low background may lead to higher difference between $< \eta >$ and C than a high background without the asymmetry. This is true for low statistics only. We see from Figs. 4 and 5 that at large λ the deviation of f from the unity is greater for the high background than for

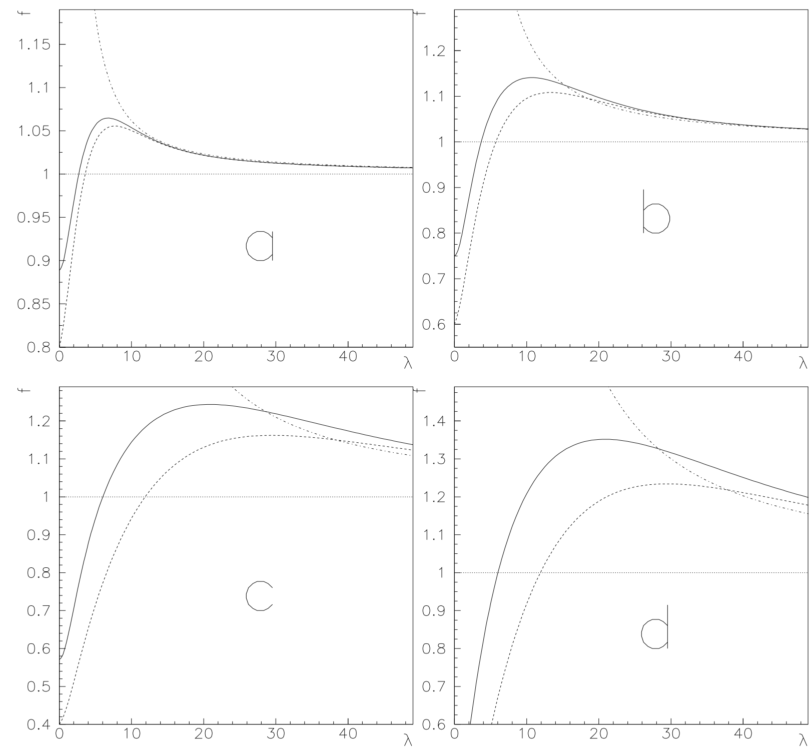

Fig. 4: Dependence of ratios $<\eta>/C$ on λ. Solid, dashed and dash-dotted curves are calculated using (76), (93) and (101), respectively. Parameters are: a) $x_0 = 0.55$, $y_0 = 0.35$, $z_0 = 0.05$, $t_0 = 0.05$, $C = 0.25$, $b/s = 0.125$ b) $x_0 = 0.5$, $y_0 = 0.3$, $z_0 = 0.1$, $t_0 = 0.1$, $C = 1/3$, $b/s = 1/3$ c) $x_0 = 0.45$, $y_0 = 0.25$, $z_0 = 0.15$, $t_0 = 0.15$, $C = 0.5$, $b/s = 0.75$ d) $x_0 = 0.45$, $y_0 = 0.25$, $z_0 = 0.1$, $t_0 = 0.2$, $C = 0.75$, $b/s = 0.75$. Parameters x_0, y_0, z_0, t_0 and λ are defined in (105), C is given by (54) and $b/s = (z+t)/(x+y-z-t)$ denotes the background to signal ratio.

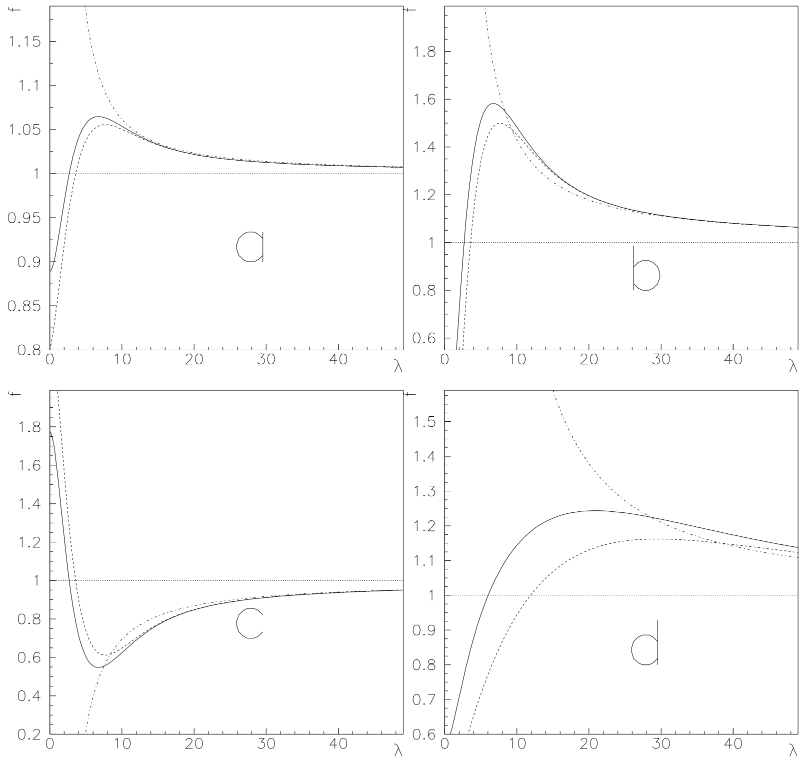

Fig. 5: Dependence of ratios $<\eta>/C$ on λ. Solid, dashed and dash-dotted curves are calculated using (76), (93) and (101), respectively. Parameters are: a) $x_0 = 0.46$, $y_0 = 0.44$, $z_0 = 0.05$, $t_0 = 0.05$, $C = 0.025$, $b/s = 0.125$ b) $x_0 = 0.45$, $y_0 = 0.45$, $z_0 = 0.04$, $t_0 = 0.06$, $C = 0.025$, $b/s = 0.125$ c) $x_0 = 0.47$, $y_0 = 0.43$, $z_0 = 0.06$, $t_0 = 0.04$, $C = 0.025$, $b/s = 0.125$ d) $x_0 = 0.355$, $y_0 = 0.345$, $z_0 = 0.15$, $t_0 = 0.15$, $C = 0.025$, $b/s = 0.75$. All notations are the same as in Fig. 4.

the low background. It is easy to see that the deviations of all curves from the unity is very appreciable. Figures 4 and 5 illustrate a usefulness of formulas (76), (93) which can be applied for estimating the difference between $< \eta >$ and the physical asymmetry C. We see from Figs. 4 and 5 that the difference between formulas (76) and (93) reaches about few per cents when $\lambda = x + y + z + t$ is about few dozens. It is easy to see also from a comparison of the curves presented in Figs. 4 and 5 that the asymptotic relation (101) predicts $< \eta >$ with rather high accuracy for $\lambda \geq 20 - 30$ even for the essential contribution of the background events when the deviation of $< \eta >$ from C is appreciable ($\sim 20\%$).

4 Conclusions

We have obtained the precise formulas for the expected values and the variances of the random variables ξ and η which correspond to the asymmetry of some reaction when the background process contribution is negligible and appreciable, respectively. Introducing the conditional probability distribution we have established that the expected value of ξ is equal to the physical asymmetry C (see (24)) if there is no background process contribution to the reaction under study. We have obtained precise formula (32) for the variance of the random variables ξ which is valid if one may ignore the background contribution. Relation (32) shows that the variance is finite, and hence ξ has nothing common with the property of the random variable with the Cauchy density function. This is the result of defining ξ in the region $p + n > 0$ and making use of the conditional probability distribution. Formula (32) predicts that the variance vanishes if the physical asymmetry squared tends to the unity. For a large number of experimental events N^{exp} the standard deviation $\delta\xi$ goes to zero as $\sqrt{1/N^{exp}}$ according to (36). The finiteness of the variance of ξ allows to utilize all the experimental statistics accumulated in many bins (for which expected values of the random variables describing the measured asymmetry are equal to each other) to get a higher statistical accuracy of the asymmetry. We may consider the random variable ζ defined by (12) which has the expected value equal to the studied asymmetry but with the variance smaller than the variance $\delta\zeta_j^2$ which corresponds to the statistical uncertainty of the asymmetry obtained from the jth bin.

As follows from (79) and (95) the expected value of η both in region (55) and (56) does not coincide with the physical asymmetry C (defined in (54)) for the process under investigation if the background contribution is not negligible. Relations (76) and (93) give the precise expressions for the expected values of the random variable η obtained with the conditional probability distributions defined in regions (56) and (55), respectively. Exact formulas (84), (83) and (76) give the expression for the variance of η in region (56). Precise relations (84), (96) and (93) are to be used for computing the variance of the random variable η defined in region (55). We conclude from these relations that the variance is finite in both cases under discussion. This important result means that we may use the statistics of the experimental events accumulated in many bins to reduce the statistical uncertainty of the measured asymmetry. We would like to stress that, nevertheless, the obtained asymmetry has a systematic uncertainty since the expected value of η deviates from the true physical asymmetry C in every bin. Formulas (79) and (95) can be applied to estimate the systematic errors due to the background contribution. The asymptotic formulas (101) and (103) for $< \eta >$ and $\delta\eta^2$ have been obtained for $x+y+z+t \gg 1$ and $(z+t)/(x+y) < 1$. This high statistics limit shows that $< \eta >$ coincides with the physical asymmetry C and the standard deviation $\delta\eta \sim \sqrt{x+y+z+t}/(x+y-z-t) \ll 1$ if conditions (100) and (102) are fulfilled.

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5 Appendix

Let us consider the asymptotic behaviour of $< \eta >$ and $\delta\eta^2$ when $x+y+z+t \to \infty$ but the ratio $Q = (z+t)/(x+y) < 1$. As has been mentioned the final results do not depend on the value of a but it is convenient to consider the limit of (67) and (76) at $a \to 1$ ($a > 1$). Since the integrands in (67) and (76) at $\phi = \theta$ and $\phi = 2\pi - \theta$ are complex conjugated functions we may integrate from 0 to π considering the real part of the integrals only. It is easy to see that the dominant contribution to the integrals in (67) and (76) comes from the regions $0 \le \phi \le \beta$ and $2\pi - \beta \le \phi \le 2\pi$ with $\beta^2 \sim (x+y+z+t)^{-1} \ll 1$ if $x+y+z+t$ is very large. Decomposing $e^{\pm i\phi}$ into power series up to terms ϕ^2 we get

$$W_+ \approx 1 - \frac{\epsilon}{\pi}\int_0^\pi d\phi \mathrm{Re}\Big\{\Big[\frac{1}{1-Q} - \frac{i\phi Q}{(1-Q)^2} - \frac{\phi^2 Q(1+Q)}{2(1-Q)^3}\Big]$$

$$\exp\{(x+y+z+t)(1-\frac{\phi^2}{2})+i\phi(x+y-z-t)\}\Big\}. \quad (106)$$

Neglecting exponentially small corrections it is possible to integrate in (106) within the limits 0 and ∞. Making use of the following asymptotic formulas valid for positive and large u and v

$$\begin{aligned} K_0(u,v) &= \int_0^\infty \exp\{-\frac{1}{2}ut^2+ivt\}dt \\ &= \frac{i}{v}\Big[1+\frac{u}{v^2}+\frac{3u^2}{v^4}+O(\frac{u^3}{v^6})\Big] , \\ K_1(u,v) &= \int_0^\infty it\exp\{-\frac{1}{2}ut^2+ivt\}dt = \\ &\quad -\frac{i}{v^2}\Big[1+\frac{3u}{v^2}+O(\frac{u^2}{v^4})\Big] , \\ K_2(u,v) &= \int_0^\infty (it)^2\exp\{-\frac{1}{2}ut^2+ivt\}dt = \frac{2i}{v^3}\Big[1+O(\frac{u}{v^2})\Big] \end{aligned} \quad (107)$$

(which can be easily derived by integration by parts) and putting (107) into (106) we get that for large $x+y+z+t$

$$W_+ = 1 \quad (108)$$

since all the integrals in (107) have no real parts. The corrections to formula (108) are exponentially small in accordance with (99).

Since a behaviour of $\ln(1-e^{-i\phi})$ at small ϕ looks like

$$\ln(1-e^{-i\phi}) = i\frac{\pi}{2}+\ln\phi-i\frac{\phi}{2}-\frac{\phi^2}{24}+... \quad (109)$$

we need the asymptotics at large positive u and v for integrals containing logarithms

$$G_0(u,v) = \int_0^\infty \ln t\exp\{-\frac{1}{2}ut^2+ivt\}dt =$$

$$\frac{i}{v}\psi(1)+\frac{iu}{v^3}\psi(3)+\frac{3iu^2}{v^5}\psi(5)$$
$$+\Big[\frac{i\pi}{2}-\ln v\Big]K_0(u,v)+O(\frac{u^3}{v^7})\ ,$$
$$G_1(u,v)=\int_0^\infty (it)\ln t\exp\{-\frac{1}{2}ut^2+ivt\}dt=$$
$$-\frac{i}{v^2}\psi(2)-\frac{3iu}{v^4}\psi(4)+\Big[\frac{i\pi}{2}-\ln v\Big]K_1(u,v)+O(\frac{u^2}{v^6})\ ,$$
$$G_2(u,v)=\int_0^\infty (it)^2\ln t\exp\{-\frac{1}{2}ut^2+ivt\}dt=$$
$$\frac{2i}{v^3}\psi(3)+\Big[\frac{i\pi}{2}-\ln v\Big]K_2(u,v)+O(\frac{u}{v^5}) \quad (110)$$

with $\psi(z)=\Gamma'(z)/\Gamma(z)$ being the digamma function. We shall obtain the expression for $G_0(u,v)$. Other formulas (110) can be derived in an analogous way. To calculate $G_0(u,v)$ we put the Frullani formula

$$\ln t=\int_0^\infty \frac{d\beta}{\beta}[e^{-\beta}-e^{-\beta t}] \quad (111)$$

into the definition of G_0 in (110) and integrate over t

$$G_0(u,v)=\int_0^\infty \frac{d\beta}{\beta}\int_0^\infty \Big[\exp\{-\frac{1}{2}ut^2+ivt\}e^{-\beta}$$
$$-\exp\{-\frac{1}{2}ut^2+i(v+i\beta)t\}\Big]dt=$$
$$\int_0^\infty \frac{d\beta}{\beta}\Big[e^{-\beta}K_0(u,v)-K_0(u,v+i\beta)\Big]\approx$$
$$\int_0^\infty \frac{d\beta}{\beta}\Big[e^{-\beta}\Big(\frac{i}{v}+\frac{iu}{v^3}+\frac{3iu^2}{v^5}\Big)$$
$$-\frac{i}{v+i\beta}-\frac{iu}{(v+i\beta)^3}-\frac{3iu^2}{(v+i\beta)^5}\Big]=$$

$$\int_0^\infty \frac{d\beta}{\beta}\Big\{\frac{i}{v}\Big[e^{-\beta} - \frac{1}{1+i\beta/v}\Big] + \frac{iu}{v^3}\Big[e^{-\beta} - \frac{1}{(1+i\beta/v)^3}\Big] + \frac{3iu^2}{v^5}\Big[e^{-\beta} - \frac{1}{(1+i\beta/v)^5}\Big]\Big\}. \quad (112)$$

One may write the integral

$$M_n = \int_0^\infty \frac{d\beta}{\beta}\Big[e^{-\beta} - \frac{1}{(1+i\beta/v)^n}\Big]$$

as the sum of the two integrals

$$M_n = \int_0^\infty \frac{d\beta}{\beta}\Big[e^{-\beta} - \frac{1}{(1+\beta)^n}\Big] + \int_0^\infty \frac{d\beta}{\beta}\Big[\frac{1}{(1+\beta)^n} - \frac{1}{(1+i\beta/v)^n}\Big] . \quad (113)$$

The first integral in (113) is equal to the digamma function due to the well known Dirichlet formula [4]. To calculate the second integral in (113) we write the chain of the equations

$$\int_0^\infty \frac{d\beta}{\beta}\Big[\frac{1}{(1+\beta)^n} - \frac{1}{(1+i\beta/v)^n}\Big]$$
$$= \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}\int_0^\infty \frac{d\beta}{\beta}\Big[\frac{1}{x+\beta} - \frac{1}{x+i\beta/v}\Big]\Big|_{x=1}$$
$$= \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}\int_0^\infty \frac{d\beta}{x}\Big[\frac{i/v}{x+i\beta/v} - \frac{1}{x+\beta}\Big]\Big|_{x=1}$$
$$= \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}\Big[\frac{1}{x}\ln\Big(\frac{i}{v}\Big)\Big]\Big|_{x=1} = i\frac{\pi}{2} - \ln v. \quad (114)$$

Putting the expressions for both integrals in (113) we get

$$M_n = \int_0^\infty \frac{d\beta}{\beta}\Big[e^{-\beta} - \frac{1}{(1+i\beta/v)^n}\Big] = \psi(n) + i\frac{\pi}{2} - \ln v\ . \quad (115)$$

Combining (115) with (112) we get the expression (110) for $G_0(u,v)$.

To compute $<\eta>$, we put (108) and (109) in (76), decompose $e^{\pm i\phi}$ into power series, integrating within limits 0 and ∞ and making use of (107), (110) and get

$$\begin{aligned}
<\eta> \approx \frac{1}{\pi}\mathrm{Re}\Big\{\int_0^\infty d\phi\Big[i\frac{\pi}{2} + \ln\phi - i\frac{\phi}{2} - \frac{\phi^2}{24}\Big] \\
\{(x-y+t-z)(1-\phi^2/2) + i\phi(x-y+z-t)\} \\
\exp\{-\frac{1}{2}(x+y+z+t)\phi^2 + i\phi(x+y-z-t)\}\Big\} \approx \\
(x-y+t-z)\Big\{\frac{1}{v} + \frac{u}{v^3} + \frac{3u^2}{v^5} + \frac{1}{v^3}\Big\} \\
-(x-y+z-t)\Big\{\frac{1}{v^2} + \frac{3u}{v^4}\Big\}\ . \qquad (116)
\end{aligned}$$

Formula (116) coincides with (101) since $u = x+y+z+t$ and $v = x+y-z-t$. We would like to note that the contributions to the real part give the terms $i\pi/2$ and $\ln\phi$ in the square brackets in (116), their contributions being equal to each other in accordance with (107) and (110).

As has been mentioned above the contribution of region $p+n < m+k$ vanishes in the limit $x+y+z+t \to \infty$, $Q = (z+t)/(x+y) < 1$. This means that we are to get the same result (116) from (93). Indeed, ϵI_0 in (93) vanishes exponentially

$$\begin{aligned}
\epsilon I_0(2\sqrt{(x+y)(z+t)}) \\
\sim \exp\{-x-y-z-t+2\sqrt{(x+y)(z+t)}\} \to 0\ . \qquad (117)
\end{aligned}$$

A comparison of (76) and (93) shows that the additional terms in (93) after the substitution $\phi \to -\phi$ (we remind that $a = 1$)

look like (relation (117) is taken into account)

$$\frac{\epsilon}{\pi}\mathrm{Re}\int_0^\infty [(x-y)e^{i\phi}+(t-z)e^{-i\phi}]$$
$$E(x+y,z+t,\phi)\ln(1-e^{i\phi})d\phi\ . \tag{118}$$

A comparison of (118) and (76) shows that the only difference between them is the difference between the logarithms. The logarithm $\ln(1-e^{i\phi})$ at small ϕ looks like

$$\ln(1-e^{i\phi}) = -i\frac{\pi}{2}+\ln\phi+i\frac{\phi}{2}-\frac{\phi^2}{24}+\dots\ . \tag{119}$$

As has been explained above only the terms $i\pi/2$ and $\ln\phi$ in the decomposition of the logarithm $\ln(1-e^{i\phi})$ contribute to the real part of the integrals in (118). But they have opposite signs in (119) (compare (119) with (109)) and hence their contributions cancel each other totally. This means that (93) does really coincide with (76) in the limit $x+y+z+t\to\infty$, $Q<1$ (which means conditions (100) and (102) to be valid).

To find an asymptotic behaviour of $<\eta^2>$ at large $x+y+z+t$ and $Q=(z+t)/(x+y)<1$ we put relation (108) in (83) and consider the limit $a\to 1$ ($a>1$). Like in calculating $<\eta>$ the dominant contributions to the integrals in (83) come from the two regions $0\le\phi\le\beta$, $2\pi-\beta\le\phi\le 2\pi$ ($\beta\ll 1$). These contributions are complex conjugate quantities. Hence we may consider the real part of the integral over ϕ in the limits 0 and π. Since the integrals are convergent very rapidly we may use the limits 0 and ∞ neglecting exponentially small corrections. To fulfill this program we have to establish the behaviour of the function

$$L_1=\int_0^\infty\frac{\alpha e^{-\alpha}}{1-\theta e^{-\alpha}}d\alpha \tag{120}$$

at small ϕ where $\theta = e^{-i\phi}$ in (120). The other functions of ϕ in (83) can be expressed through L_1. Indeed, it is obvious that

$$L_2 = \int_0^\infty \frac{\alpha e^{-2\alpha}}{1-\theta e^{-\alpha}} d\alpha = \frac{L_1}{\theta} - \frac{1}{\theta} \, . \tag{121}$$

Applying the new variable $t = e^{-\alpha}$ and integrating several times by parts we get

$$\begin{aligned}
L_1 &= -\int_0^1 \frac{\ln t}{1-t\theta} dt \\
&= \frac{\ln t}{\theta} \ln(1-t\theta)\Big|_0^1 - \int_0^1 \frac{dt}{t\theta} \ln(1-t\theta) = \\
&\frac{1}{t\theta^2}[(1-t\theta)\ln(1-t\theta)+t\theta]\Big|_0^1 \\
&+ \int_0^1 \frac{dt}{t^2\theta^2}[(1-t\theta)\ln(1-t\theta)+t\theta] = \\
&\frac{1}{\theta^2}[(1-\theta)\ln(1-\theta)+\theta] \\
&-\frac{1}{t^2\theta^3}\Big[\frac{(1-t\theta)^2}{2}\ln(1-t\theta) - \frac{3}{4}(1-t\theta)^2 - t\theta + \frac{3}{4}\Big]\Big|_0^1 \\
&-\frac{2}{\theta^3}\int_0^1 \Big[\frac{(1-t\theta)^2}{2}\ln(1-t\theta) - \frac{3}{4}(1-t\theta)^2 - t\theta + \frac{3}{4}\Big]\frac{dt}{t^3} = \\
&L_1^a + L_1^b + L_1^c.
\end{aligned} \tag{122}$$

It is easy to see that at small ϕ

$$\begin{aligned}
L_1^a &= \frac{1}{\theta^2}[(1-\theta)\ln(1-\theta)+\theta] \\
&\approx 1 + i\phi + (i\phi - \frac{3}{2}\phi^2)(\ln\phi + i\frac{\pi}{2}) + \ldots \, , \\
L_1^b &= -\frac{1}{\theta^3}\Big[\frac{(1-\theta)^2}{2}\ln(1-\theta) - \frac{3}{4}(1-\theta)^2 - \theta + \frac{3}{4}\Big] \\
&\approx \frac{1}{4} - \frac{i}{4}\phi + \frac{5}{8}\phi^2 + \frac{\phi^2}{2}[\ln\phi + i\frac{\pi}{2}] + \ldots
\end{aligned} \tag{123}$$

and hence the derivatives $\partial L_1^a/\partial\phi$, $\partial^2 L_1^a/\partial\phi^2$, $\partial^2 L_1^b/\partial\phi^2$ do not exist at $\phi = 0$ since expressions (123) contain $\ln\phi$. One may check that $\partial L_1^c/\partial\phi$, $\partial^2 L_1^c/\partial\phi^2$ are finite at $\phi = 0$ which means that L_1^c may contain terms $\phi^n \ln\phi$ with $n \geq 3$ only. Differentiating L_1^c over ϕ it can be easily obtained that

$$L_1^c =$$
$$-\frac{2}{\theta^3}\int_0^1 \Big[\frac{(1-t\theta)^2}{2}\ln(1-t\theta) - \frac{3}{4}(1-t\theta)^2 - t\theta + \frac{3}{4}\Big]\frac{dt}{t^3}$$
$$\approx -\frac{5}{4} + \zeta(2) + i\phi[-\frac{7}{4} + \zeta(2)] + \frac{\phi^2}{2}[-\frac{3}{4} + \zeta(2)] + \ldots \quad (124)$$

where $\zeta(2) = \pi^2/6$ and $\zeta(z)$ denotes the Riemann ζ-function. Putting (123) and (124) into (122) we get

$$L_1 = \zeta(2) + i\phi[\zeta(2) - 1] + \frac{\phi^2}{2}[\zeta(2) + \frac{1}{2}]$$
$$+(i\phi - \phi^2)(\ln\phi + i\frac{\pi}{2}) + \ldots \quad . \quad (125)$$

The term $\epsilon I_1(2\sqrt{(x+y)(z+t)})$ in (83) is exponentially small owing to (98). Decomposing $e^{\pm im\phi}$ in (83) into power series, putting them and (125), (121), (108) into (83) and integrating over ϕ from 0 to ∞ one gets

$$< \eta^2 >= (x - y + t - z)^2(\frac{1}{v^2} + \frac{3u}{v^4}) + \frac{u}{v^2} + \frac{3u^2}{v^4}$$
$$-\frac{2}{v^2} - \frac{4}{v^3}[(x-y)^2 - (t-z)^2] \quad (126)$$

with $u = x+y+z+t$ and $v = x+y-z-t$. Substitution of (126) and (101) into (84) gives the final formula (103) for the variance $\delta\eta^2$ if conditions (100) and (102) are valid.

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