The IM factorization and HE Amplitude

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"Hadron Structure and QCD", St. Petersburg
May 18 - 22 , 2004

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Goals:

1. **New interpretation of IM factorization.**
2. **Asymptotic behavior of the scattering amplitude.**

Topics:

- The Iancu-Mueller factorization and the Reggeon-Like diagram technique;
- Generating Functional;
- The Iancu-Mueller factorization in a toy model;
- High energy asymptotic behavior of the scattering amplitude for fixed and running α_S ; [A new solution to BK equation]
- Amplitude in the saturation region;

IM Factorization

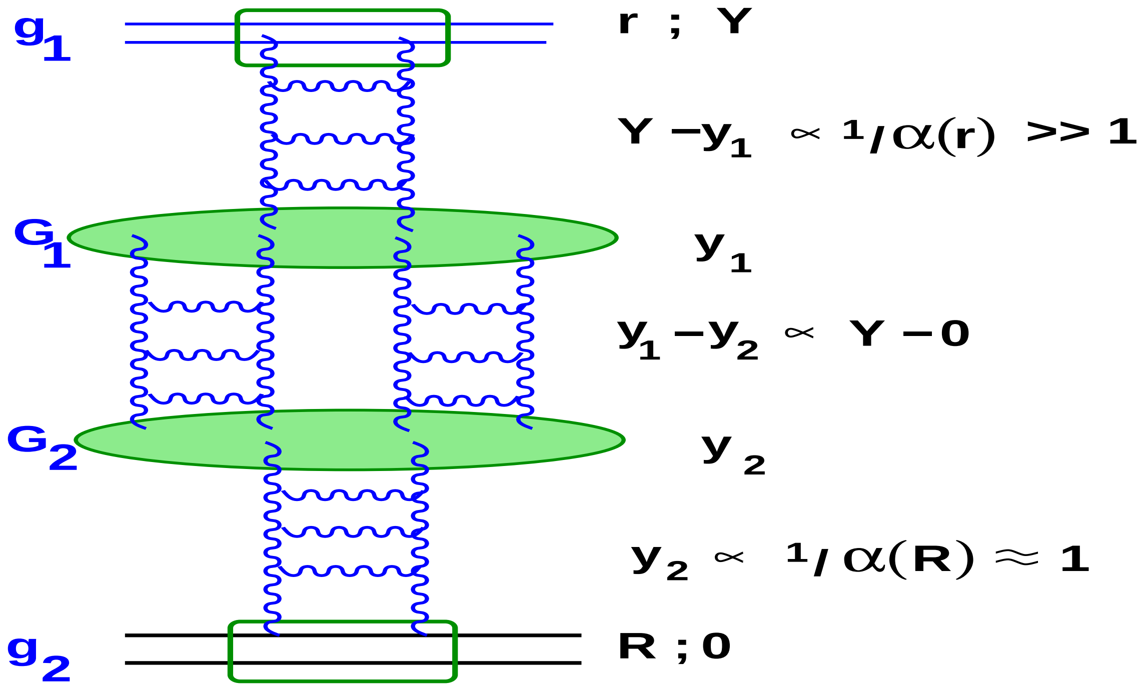
The Iancu-Mueller approach takes into account fluctuations in the partonic wave function of the fast moving particle which were neglected in the non-linear equation.

Fluctuations in the partonic wave function = Reggeon-Like diagram tehnique.

Toy Model:

- **Any dependence on the size of the interacting dipoles, is neglected**
- **The BFKL Pomeron intercept:** $\Delta \propto \alpha_S$
- $G_1 = g_1 = \Delta$; $G_2 = g_2 = \frac{\Delta}{N_c^2}$
- **Correct order of** α_s **and** N_c **in pQCD**

The first enhanced diagram :



- **Reggeon-Like Calculation:**

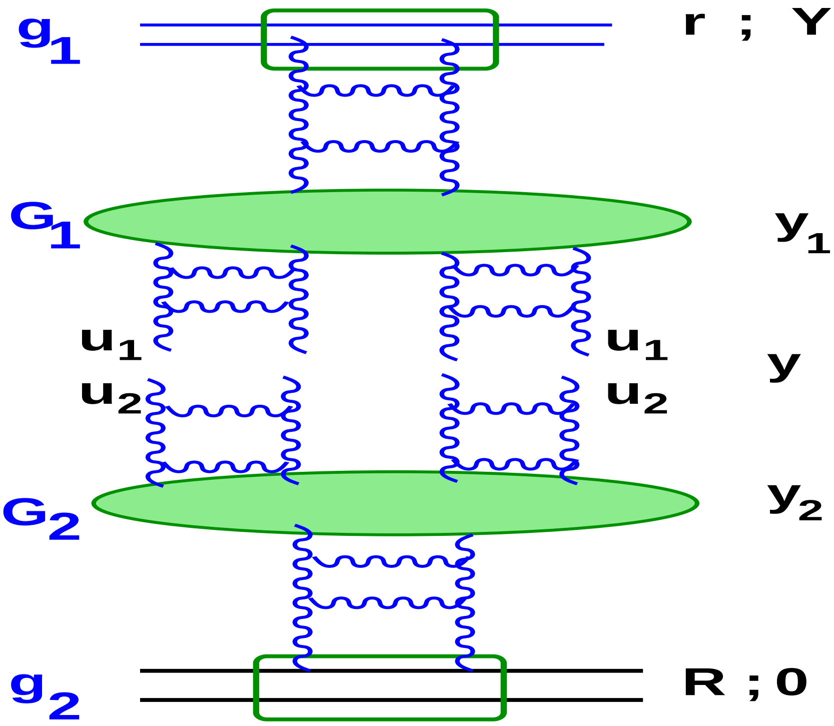
$$A = (-1) g_1 g_2 G_1 G_2 \cdot e^{\Delta Y} \left(\frac{1}{\Delta^2} \{ e^{\Delta Y} - 1 \} - \frac{Y}{\Delta} \right)$$

$$= (-1) \frac{\Delta^2}{N_c^4} e^{\Delta Y} \left(\{ e^{\Delta Y} - 1 \} - \Delta Y \right)$$

$$\longrightarrow \boxed{(-1) \frac{\Delta^2}{N_c^4} e^{2 \Delta Y}}$$ (for large N_c)

- $N_c^2 \gg \frac{1}{N_c^2} e^{\Delta Y} \geq 1$

The IM factorization for the first enhanced diagram for dipole-dipole scattering.



- **Generating Function Calculation:**

- $$Z(y,u) \;=\; \sum_n \; P_n(Y-y)\; u^n$$
- $$A(Y-y,\gamma) = 1 - Z(Y-y, u \equiv 1+\gamma) \\ = (-1)\gamma^2 e^{\Delta(Y-y)}\left(e^{\Delta(Y-y)} - 1\right)$$
- $$A = \Delta^2(-1)A(Y-y,\gamma=1)A(y,\gamma=\frac{1}{N_c^2})$$

$$\longrightarrow \quad \boxed{(-1)\frac{\Delta^2}{N_c^4}e^{2\Delta Y}}$$

Generating Functional

Problem: *separation of the high energy part of 'fan' diagrams from the low energy part.*
Solution: *generating functional.*

(A.Mueller 1995)

$$Z\left(Y-y,r;[u_i]\right)\;\equiv$$
$$\sum_n \int P_n(Y-y,r;r_1,b_1\ldots r_n,b_n)\prod_{i=1}^{n} u(\vec{r}_i,\vec{b}_i)d^2r_i d^2b_i$$

Initial conditions for the functional:

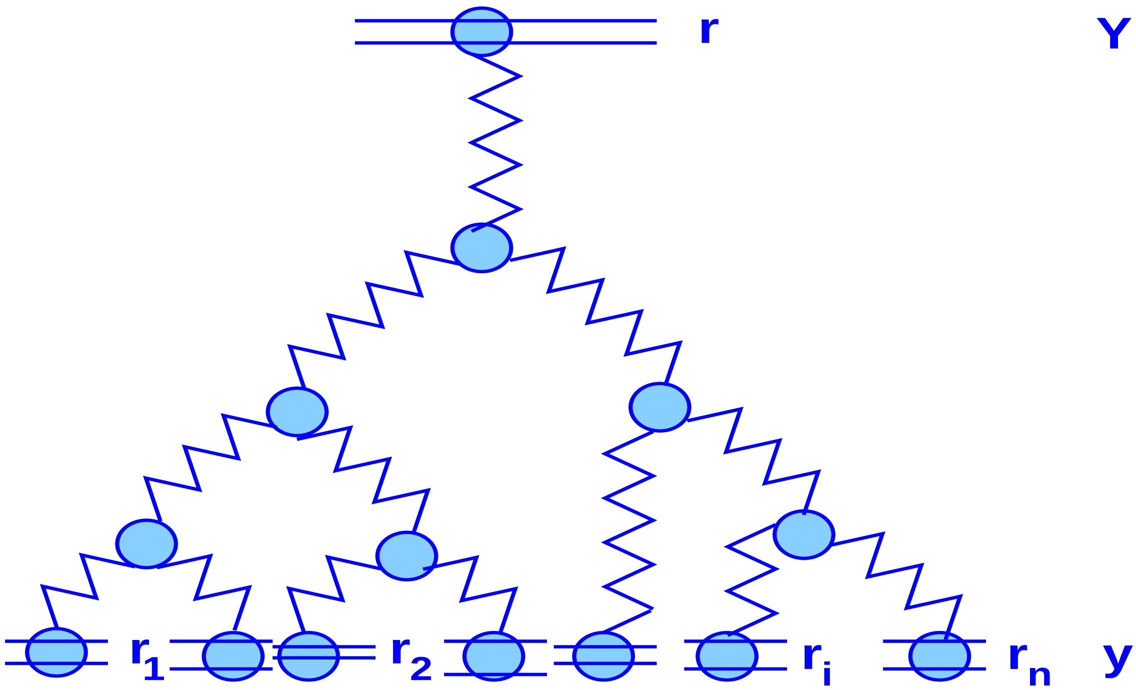
1 *At* $y=Y,\;P_1=\delta^{(2)}(\vec{r}-\vec{r}_1),\;P_{n>1}=0$

$$Z\left(Y-y=0,\;r;\;[u_i]\right)\;=\;u(r)$$

2 *At* $u_i=1$

$$Z\left(Y-y,\;r;\;[u_i=1]\right)\;=\;1$$

Generating functional sums 'fan' diagrams $\longrightarrow$ Simple linear functional equation.



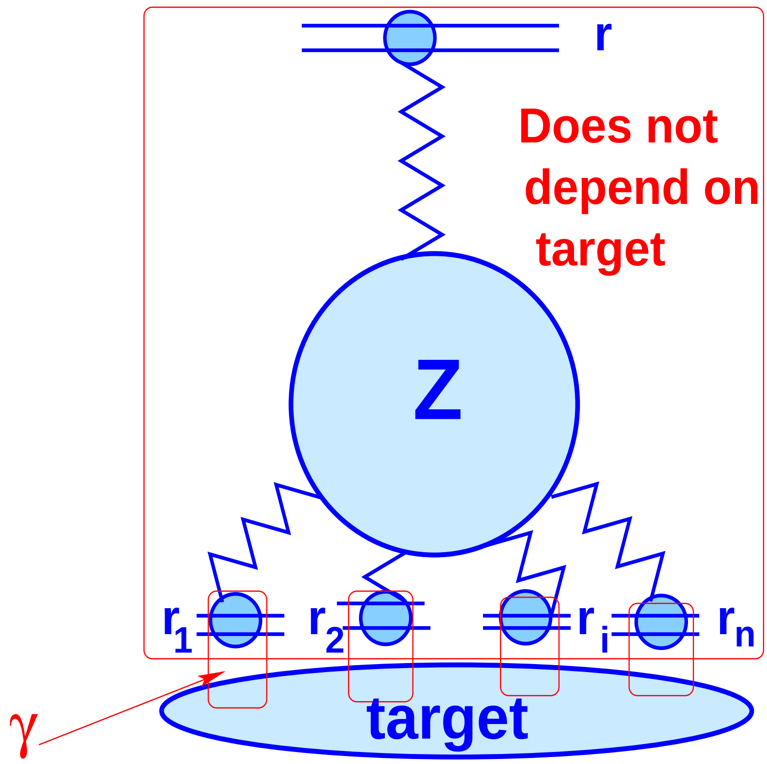
- $\frac{\partial Z(Y-y,r;[u_i])}{\partial \bar{\alpha}_S\, y} = -\int d^2 r_i u(r_i)\omega(r_i)\frac{\delta}{\delta u_i} Z\left(Y-y,r;[u_i]\right)$
$+\int d^2 r_i d^2 r' u(r_i) u(\vec{r}_i - \vec{r}')\frac{r'^2}{r_i^2(\vec{r}_i-\vec{r}')^2}\frac{\delta}{\delta u(r')} Z\left(Y-y,r;[u(r'),u_i]\right)$

- The first term = probability for the BFKL Pomeron to propagate $y \longrightarrow y + dy$ (without decay).

- The second term = possibility for decay of one dipole to two dipoles (triple BFKL Pomeron vertex).

Solution to GF equation can be written as a function of a single variable $u(y)$.

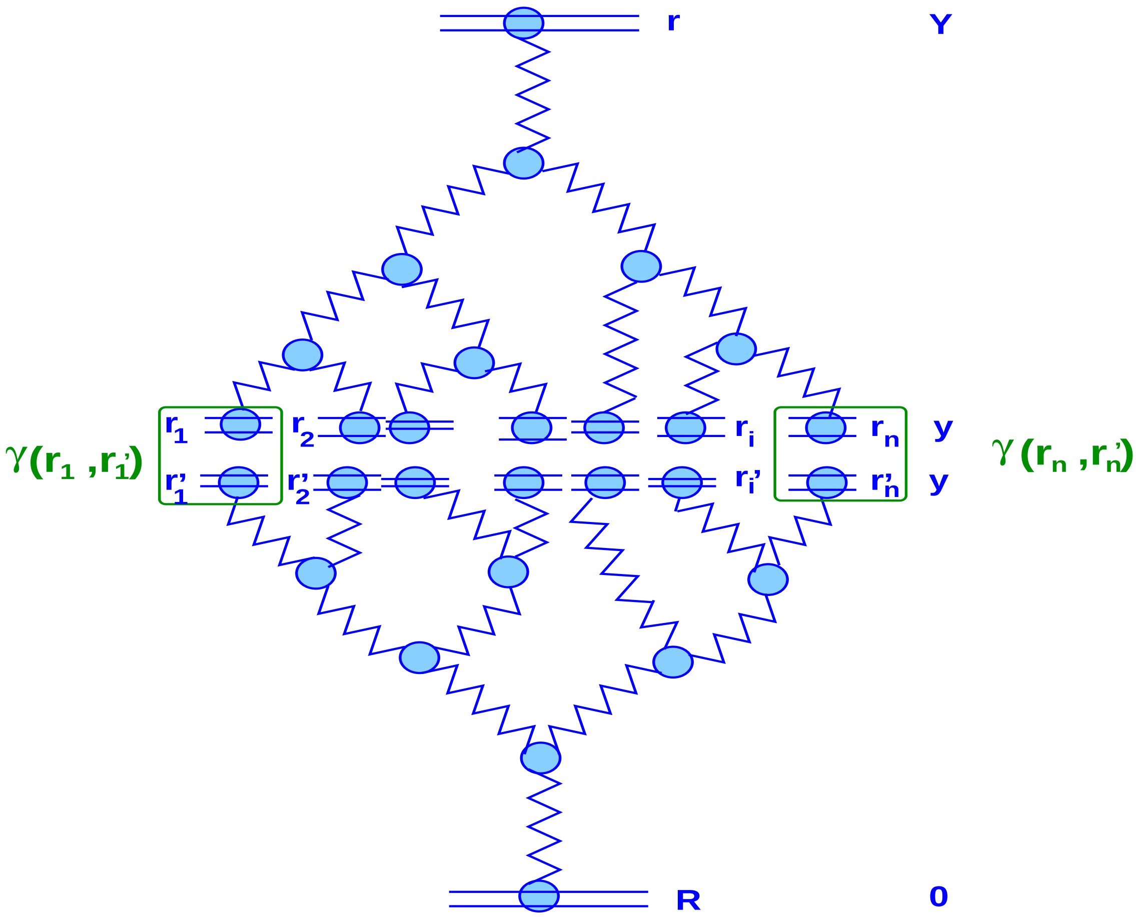
$$\frac{dZ(Y-y,r;[u_i])}{d\bar{\alpha}_S\,Y} = -\omega(r)\;Z\,(Y-y,r;[u_i])$$
$$+\int d^2r'\frac{r^2}{r'^2(\vec{r}-\vec{r}')^2}\;Z\,(Y-y,r';[u_i])\,Z\,(Y-y,\vec{r}-\vec{r}');[u_i])$$



- **If all produced dipoles interact with the target independently (without correlations):**

$$N\,(Y,r;[\gamma(r_i,b_i)]) = 1 - Z\,(Y,r,b_t;[\gamma(r_i,b_i)+1])$$

The IM factorization and enhanced diagrams.



Summing over all possible numbers of the interacting dipoles:

- $$N(r, R, Y; b) = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} d^2 b_i d^2 b'_i \prod_{i=1}^{n} d^2 r_i d^2 r'_i \tilde{\gamma}(r_i, b - b_i; r'_i, b'_i) \cdot N_n(r, Y - y; r_1, b - b_1 \dots r_n, b - b_n) \cdot N_n(R, y; r'_1, b'_1 \dots r'_n, b'_n)$$

Where:

- N_n **is the general term of expansion of the amplitude:**

$$N_n\left(r,Y,b,r_1,b_1\dots r_n,b_n\right)= \frac{1}{n!}\prod_{i=1}^{n}\frac{\delta}{\delta\gamma_i}\left(1-Z\left(Y,r;\left[\gamma(r_i,b_i)+1\right]\right)\right)|_{\gamma_i=0}$$

- $\tilde{\gamma}(r_i,r'_i)$ **is the amplitude of the dipole-target interactions at low energies:**

$$\tilde{\gamma}(b_i,b'_i;r_i,r'_i)= (-1)\;\delta^{(2)}(r_i-r'_i)\;\delta^{(2)}(b'_i-b_i)\;\frac{\bar{\alpha}_s^2\pi^3}{N_c^2}\frac{\nu^2}{(\nu^2+\frac{1}{4})^2}\frac{1}{r_i^2}$$

- **For single BFKL Pomeron exchange:**

$$N\left(Y-y,r,q;\left[\gamma(r_1)\right]\right)= \int d^2r_1 N^{BFKL}\left(Y-y,r,r_1;q\right)\gamma(r_1)$$

$$N^{BFKL}\left(Y,r,R;q\right)= -\int d^2r_1 d^2r'_1 N\left(Y-y,r,b;\left[\gamma(r_1)\right]\right)N\left(y,R,q;\left[\gamma(r'_1)\right]\right)$$

1. $\gamma(r_1)=\nu$ *and* $\gamma(r'_1)=\tilde{\gamma}(q;r_1,r'_1)/\nu$
2. *completeness relation*

IM Factorization in Toy Model

Toy Model :

- **Probability for the dipole to decay in two dipoles is a constant (ω_0);**
- **The fact, that we have dipoles of different sizes, is neglect;**

Generating Functional reduces to:

$$Z(Y - y, u) \; = \; \sum_{n=1} P_n \, u^n$$

Initial and Boundary conditions:

- **At** $y = Y$: $Z(Y - y = 0, u) = u$
- **At** $u = 1$: $Z(Y - y, u = 1) = 1$

$$-\frac{\partial Z(y,u)}{\partial y} \; = \; -\omega_0 \, (\; u(1 - u) \;) \, \frac{\partial Z(y,u)}{\partial u}$$

$$Z(Y - y, u) = \frac{u}{1 + (e^{\omega_0 (Y - y)} - 1)(1 - u)}$$

Amplitude in a Toy Model:

$$N(y,u) \;=\; 1 \;-\; Z(y,u)$$

$$N(Y-y,\gamma) \;=\; -\;\frac{\gamma e^{\omega_0(Y-y)}}{1+\gamma\left(e^{\omega_0(Y-y)}\,-\,1\right)}$$

Our suggestion for IM factorization:

$$N(Y) = \frac{1}{2\pi i}\oint\frac{d\nu}{\nu}N(Y-y,\nu\cdot\gamma_{SM})N(y,\frac{1}{\nu})$$

where $\gamma_{SM} \;=\; (-1)\,\frac{\alpha_s^2\pi^3}{N_c^2}$

Amplitude:

$$N(Y) \;=\; \frac{\frac{\alpha_s^2\pi^3}{N_c^2}e^{\omega_0 Y}}{1+\frac{\alpha_s^2\pi^3}{N_c^2}\left(e^{\omega_0(Y-y)}\,-\,1\right)(e^{\omega_0 y}\,-\,1)}$$

At $Y \gg 1$:

$$N(Y) \to 1 - \frac{\alpha_s^2\pi^3}{N_c^2}e^{-\omega_0 Y} \;+\; O\left(\frac{\alpha_s^2\pi^3}{N_c^2}e^{-\omega_0 Y/2}\right)$$

High Energy Asymptotic

General BFKL kernel:

$$\omega(\nu) \equiv \frac{\alpha_S N_c}{\pi} \chi(\gamma) = \frac{\alpha_S N_c}{\pi} \left(2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\right)$$

Where, $\psi(f) = d \ln \Gamma(f)/df$

Simplified BFKL kernel:

$$\omega(\gamma) = \frac{\alpha_S N_c}{\pi} \begin{cases} \frac{1}{\gamma} & \text{for } r^2 Q_s^2 < 1 \quad (\alpha_S \ln(r^2\Lambda^2) \ln(1/x))^n \\ \\ \frac{1}{1-\gamma} & \text{for } r^2 Q_s^2 > 1 \quad (\alpha_S \ln(r^2 Q_s^2))^n \end{cases}$$

Simplified BFKL Kernel.

Fixed α_s:

- At $r_t \, < \, 1/Q_s(Y-y,b)$

$$N\left(Y-y,r_1,b;[\gamma(r_i)]\right) = \int d^2 r_i \gamma(r_i) N^{BFKL}((Y-y,r_1,r_i,b)$$

- At $r_t \, > \, 1/Q_s(Y-y,b)$

$$N\left(Y-y,r_1,b;[\gamma(r_i)]\right) = 1 \, - \, e^{-\frac{1}{2}\left(\frac{z}{2} + \Phi[\gamma(r_i)]\right)^2}$$

- $\Phi([\gamma(r_i)]) \, \equiv \, \ln\left(\int d^2 r_i e^{-\frac{1}{2}\ln(1/r_i^2)}\gamma(r_i)\right)$

Matching of two solutions:

$$N\left(r,R,Y;b\right) \, = \, 1 \, - \, e^{-\frac{1}{16}z^2(Y,r,R)}$$

- $z(Y,r,R) \, = \, 4\,\bar{\alpha}_s\,Y \, - \, \ln(R^2/r^2)$

Running α_s (short distances):

- **At** $r_t \;>\; 1/Q_s(Y-y,b)$

1. α_s **is frozen at** $r^2 = \frac{1}{Q_s^2}$**.**

$N\,(Y,r,R;b) \;=\; 1 \;-\; e^{-\frac{1}{8}z^2}$ **= BK eq.**

- $z = \frac{8N_c}{b}\left(\sqrt{Y}\right) - \ln(R^2/r^2)$

2. α_s **depends on the size of produced dipole.**

$$\alpha_s \int \frac{r^2}{r'^2\,(\vec{r}-\vec{r}')^2}$$

$$\rightarrow \pi \int_{1/Q_s^2}^{r^2} \alpha_S(r')\frac{dr'^2}{r'^2} + \pi \int_{1/Q_s^2}^{r^2} \alpha_S(|\vec{r}-\vec{r}'|)\frac{d(\vec{r}-\vec{r}')^2}{(\vec{r}-\vec{r}')^2}$$

- $N\,(r,R,Y;b) \;=\; 1 \;-\; e^{-\phi(Y-y,r)-\phi(y,R)}$
- $\phi(Y-y,r)+\phi(y,R) \longrightarrow \phi(Y,r)+\frac{\alpha_S(\xi)}{2\pi}\left(-\xi_s^2(y_{min})+\xi^2(R)\right)$
- $\xi_s(Y)=\ln(Q_s^2(Y)/\Lambda), \qquad \xi(r)=\ln(1/(r^2\Lambda^2))$
- $\xi_s^2(y_{min})=\xi^2(r)\left(\frac{\xi_s^2(Y)}{\xi^2(R)}\right)^{\alpha_S(r)/\alpha_S(R)}$

Full BFKL Kernel.

Non-Linear equation in mixed representation:

- $\frac{\partial \tilde{N}(k,y;b)}{\partial y} \;=\; \bar{\alpha}_s \left(\chi(\hat{\gamma}(\xi)) \tilde{N}(k,y;b) \;-\; \tilde{N}^2(k,y;b) \right)$

- $\chi(\hat{\gamma}(\xi)) = 1 + \frac{\partial}{\partial \xi}, \quad \xi = \ln(k^2 k'^2 b^4)$

Ansatz for solution:

- $\tilde{N}(z) \;=\; \frac{1}{2} \int^z dz' \left(1 \;-\; e^{-\phi(z')} \right)$

- ϕ **is a smooth function:** $\phi_{zz} \ll \phi_z \phi_z$

$\Longrightarrow \quad \frac{d^n}{(dz)^n} e^{-\phi(z)} = (-\phi_z)^n e^{-\phi(z)}$

Substituting

$$\bar{\alpha}_s \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} \frac{d^2 \tilde{N}(z)}{(dz)^2} = \bar{\alpha}_s \left([f \chi(1 - \frac{d}{dz}) - 1] \tilde{N}(z) + \tilde{N}(z) e^{-\phi} \right)$$

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(dz)^2} \;=\; \left(1 - e^{-\phi(z)}\right) \;-\; \frac{dL(\phi_z)}{d\phi_z}\,\frac{d^2\phi}{(dz)^2}$$

- $$L(\phi_z) \;=\; \frac{\phi_z\,\chi(1-\phi_z) - 1}{\phi_z}$$

For $\phi \gg 1$

$$\frac{1}{(1-\phi_z)^2}\,\frac{d^2\phi}{(dz)^2} \;=\; 1$$

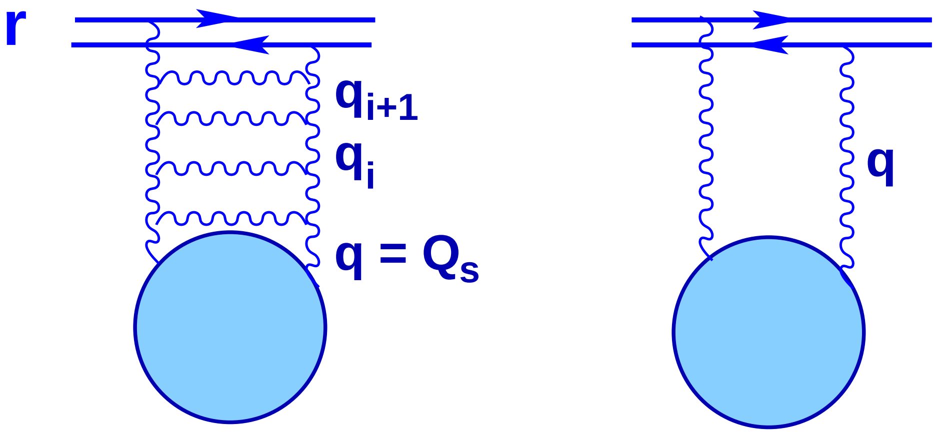
At large z: $\phi(z) \;=\; z \;-\; \ln z$

In coordinate representation:

$$\tilde{N}\,(k, y; b) \;=\; \tilde{N}\,(z) \;=\; \frac{1}{2}\int^{z} dz' N(z')$$

- $N\left(r^2 Q_s^2(y,b)\right) \;=\; 1 \;-\; e^{-z(r)\,+\,\ln z(r)}$
- $z(r) = \ln(Q_s^2\, r^2) = \bar{\alpha}_s \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}(y - y_0) - \ln(1/r^2\Lambda^2)$

Physical picture for the result.



$Q_s > q_i > q_{i+1} > 1/r$ $\qquad$ $Q_s > q > 1/r$

$e^{-c\,z^2}$ = probability for dipole to pass the target without any emission of gluons

$e^{-z + \ln z}$ = probability to have no elastic scattering

High energy amplitude in the saturation region.

Fixed α_S :

$$N\,(r, R, Y; b) = 1 - e^{-z(Y,r,R) + 2\ln\left(\frac{1}{2}z(Y,r,R)\right)}$$

- $z(Y, r, R) = 4\bar{\alpha}_s Y - \ln(R^2/r^2)$

$N\,(r, R, Y; b) \neq$ solution to BK equation !

Running α_S :

$$N\,(r, R, Y; b) = 1 - e^{\ln(r^2\,Q_s^2) - \ln\left(\ln\left(\frac{\ln(Q_s^2/\Lambda)}{\ln(1/(r^2\Lambda^2))}\right)\right)}$$

$N\,(r, R, Y; b) =$ solution to BK equation !

Results:

- The IM factorization is closely related to sum of enhanced diagrams;
- New solution of non-linear evolution equation in saturation region;
- The simple formula for the dipole-dipole amplitude, which sums enhanced diagrams and manifests the IM factorization;