Quantum mechanics of radiofrequency-driven coherent beam oscillations in storage rings

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Sequel to

Collective oscillations of a stored deuteron beam close to the quantum limit

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By all evidence the first look into the quantum regime of transversal beam oscillations

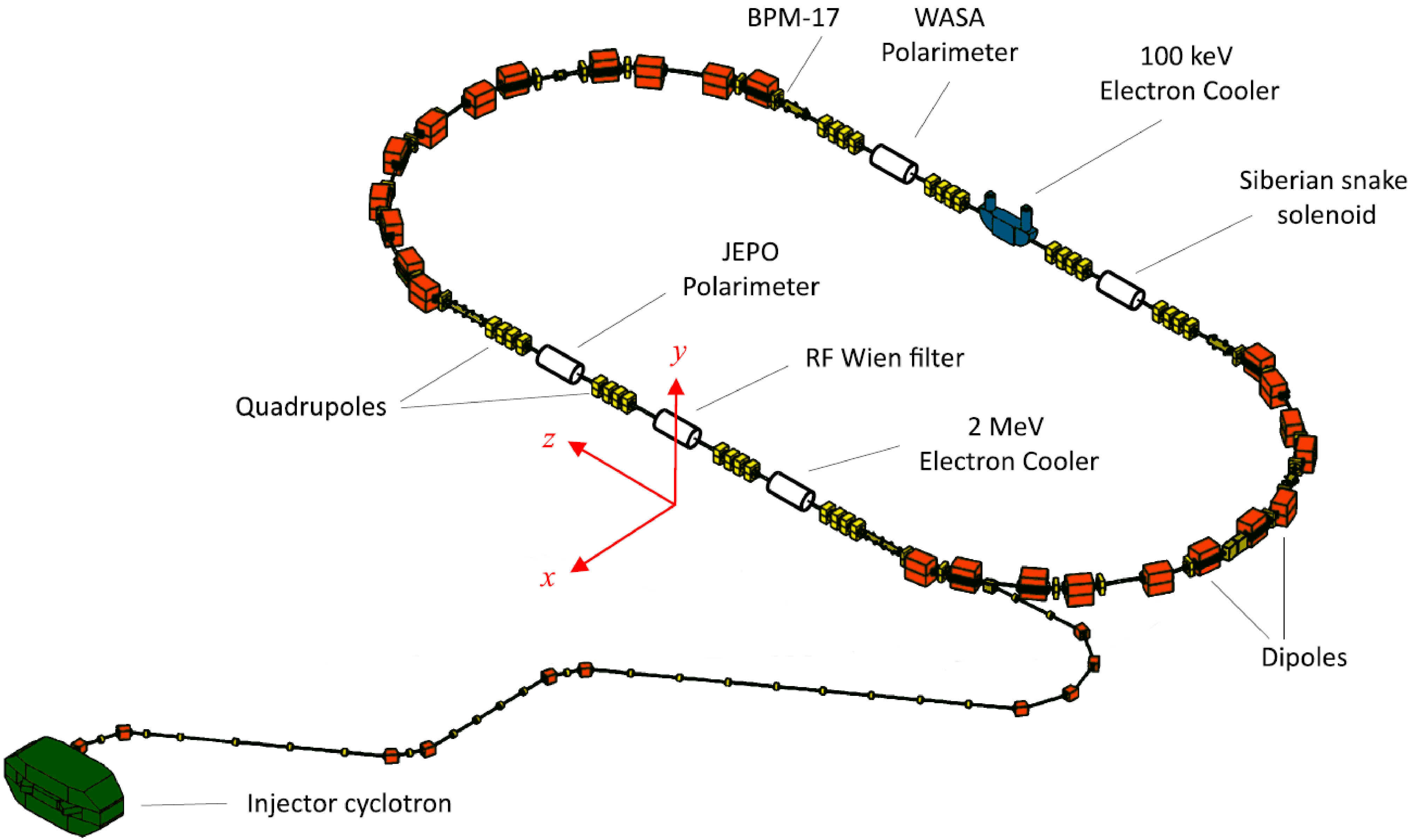
Smallest measured coherent oscillation amplitude 1.06 +/- 0.52 micrometer

Zero-point betatron oscillations in the confining harmonic oscillator potential Q = 41 nanometer

Still in the classical mechanics domain

What if the measured amplitude had been way below Q ?

How to treat sub-quantum beam oscillations in the picometer domain of ultimate pEDM storage rings ?



BPM-17
WASA
Polarimeter
100 keV
Electron Cooler
Siberian snake
solenoid
JEPO
Polarimeter
y
RF Wien filter
Quadrupoles
z
2 MeV
Electron Cooler
x
Dipoles
Injector cyclotron

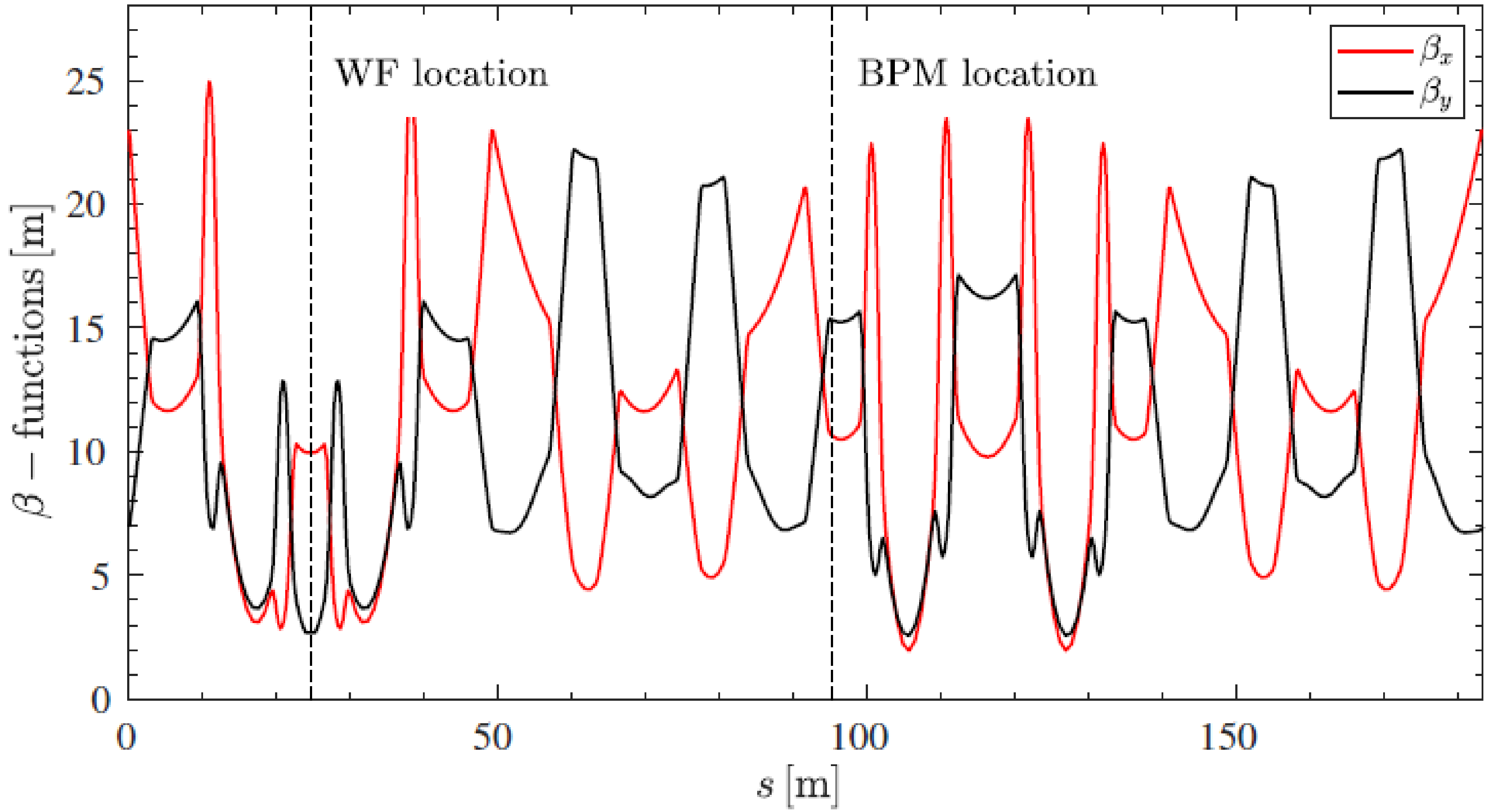
Beams are confined on steady orbits by quadrupoles

- Concurrent vertical focusing & radial defocusing and vice versa
- Strong focusing synchrotrons: doublets of 90^o rotated quadrupoles ➔ Galileo refractor, net focusing in both directions
- All electric storage ring: radial holding E-field at magic energy to cancel MDM rotation
- Potential sensitivity to EDM ~ 10^{-15} MDM
- All electric ring: possibility of concurrent clockwise and anti-closckwise beams on identical orbits → cancellation of major systematics
- It is imperative to keep the identity of two orbits at 5 picometer scale
- Table top stand expts suggest that such a sensitivity is not precluded

Idle betatron motion:

$$y(t) = y(0)\sqrt{\frac{\beta_y(t)}{\beta_y(0)}}\cos\left[\psi_y(t)\right]$$

$$\psi_y(t+T) - \psi_y(t) = \omega_y T = 2\pi\nu_y$$



All-electric frozen-spin proton EDM ring:

radial magnetic fields are no go → monitor vertical separation of counter-rotating beams to about 5 pm accuracy

JEDI/CPEDM Collab., CERN Yellow Report

Emergence of picometers from the beam displacement by the Earth gravity

Relate the oscillator spring constant to the betatron frequency

$$\Delta y \approx \frac{(2\gamma^2 - 1)\,|g_\oplus|}{\gamma^2 \nu_y^2\, \omega_{\rm rev}^2}$$

Electric focusing, identical displacements of CW and CCW beams: $\Delta y_E \approx 13\,\mathrm{pm}$

Hybrid ring with magnetic focusing $\Delta y_B \approx 1.3\,\mathrm{pm}$

Ineliminable splitting of CW and CCW trajectories ~ $2.6\,\mathrm{pm}$

Classical mechanics of stroboscopic mismatched RF Wien filter driven collective betatron oscillations

Oscillator variable $z = y - iv_y/\omega_y$

One-pass kick in the RF WF $\Delta v_y(n) = \dfrac{F_y(n)\Delta t}{\gamma m} = -\zeta\omega_y \cos(n\,\omega_{\rm WF}T)$

Master equation for stroboscopic evolution $z(n) = z(n-1)\exp(i\omega_y T) - \dfrac{i}{\omega_y}\Delta v_y(n)$

Solution for RF driven oscillations

$$z(n) = -\frac{i}{\omega_y}\exp(i\omega_y nT)\sum_{k=1}^{n}\Delta v_y(k)\exp(-i\omega_y kT) = \frac{i\zeta}{2}\,\frac{\exp(in\omega_y T) - \exp(in\,\omega_{\rm WF}T)}{\exp[i(\omega_y - \omega_{\rm WF})T] - 1} + \{\omega_{\rm WF} \to -\omega_{\rm WF}\}\,.$$

Classical mechanics of RF driven collective betatron oscillations

Isolate the RF driven component by on-line Fourier analysis

$$y_{\rm WF}(n) = y(n) = {\rm Re}\, z(n) = -\xi_y \cos(n\,\omega_{\rm WF} T)$$

$$\xi_y = \frac{\zeta}{2} \cdot \frac{\sin(2\pi\nu_y)}{\cos(2\pi\nu_{\rm WF}) - \cos(2\pi\nu_y)}$$

Resonance at $\nu_{\rm WF} = \nu_y$.

$$y^{\rm res}(n) = -\tfrac{\zeta}{2}(n-1)\sin(n\omega_y T)$$

Coherent oscillation amplitudes are identical for all particles in the bunch irrespective of their individual idle betatron oscillations

Collective signal of 10^9 deuterons in the bunch

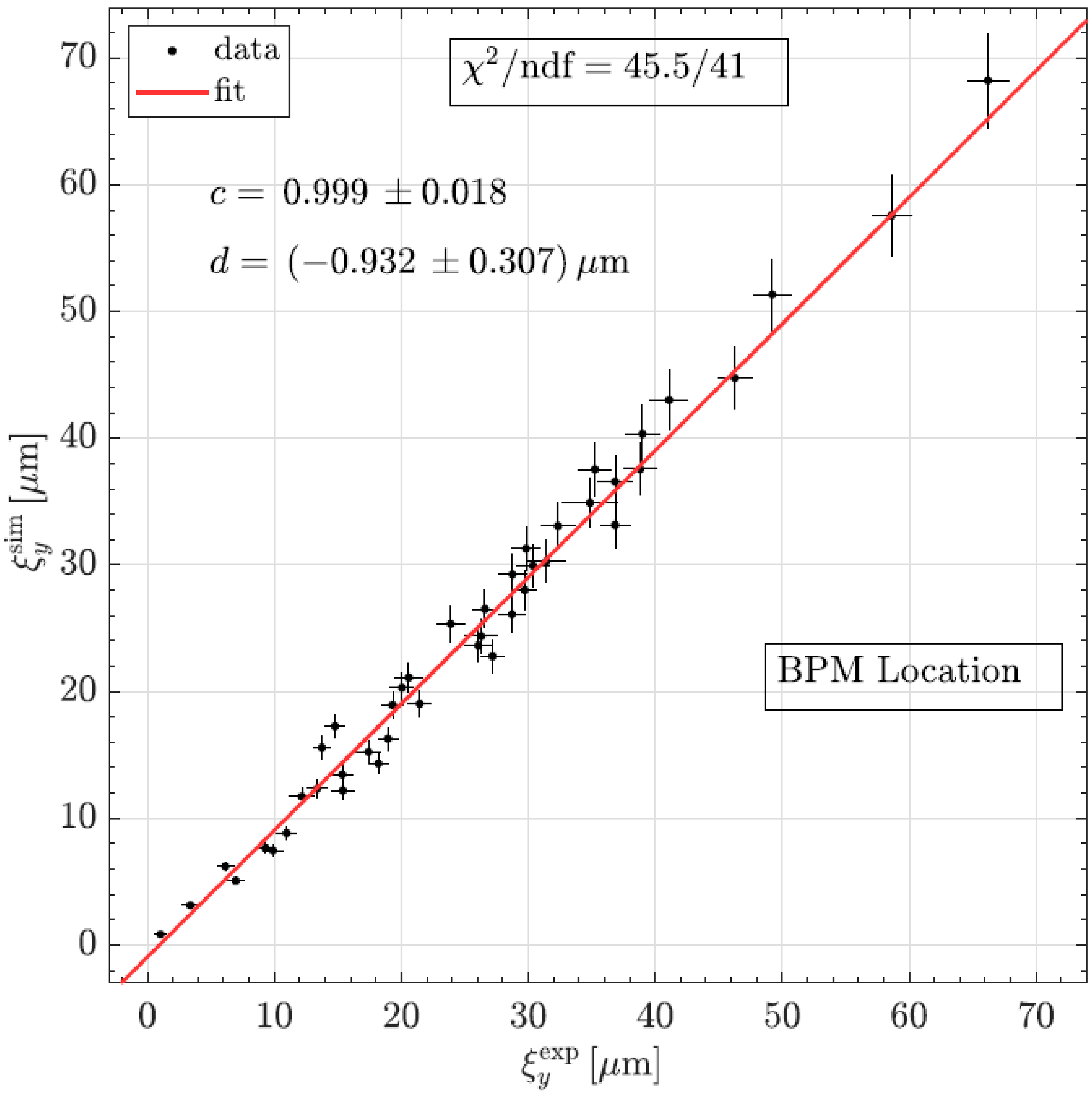
Classical mechanics of RF driven collective betatron oscillations cont'd

Idle rotation is perturbed by

- scattering off the residual gas (mostly hydrogen)
- Intrabeam scattering
- Typical lifetimes for 10^9 particles in a bunch $\tau > 10^3$ s

$$z(n) = -\frac{i}{\omega_y} \exp(i\omega_y nT) \sum_{k=1}^{n} \Delta v_y(k) \exp(-i\omega_y kT)$$

- Rare random kicks increase the beam emittance but do not affect the RF driven oscillations



$$\xi_y^{\min}\Big|_{\mathrm{BPM}} = (1.08 \pm 0.52)\,\mu\mathrm{m}$$

$$\xi_y^{\min}\Big|_{\mathrm{WF}} = (0.45 \pm 0.22)\,\mu\mathrm{m}$$

Beam size ~ 1 mm

Zero-point oscillations

$$Q^2 = \frac{\hbar}{m\gamma\omega_y}$$

$$Q \approx 41\,\mathrm{nm}\,.$$

What if the observed amplitude were smaller than Q ?

Quantum mechanics of coherent oscillations

$$i\frac{d}{dt}\Psi(t) = \{H_0 + V(t)\}\,\Psi(t)$$

Perturbative potential in terms of the HO creation and annihilation operators

$$V(t) = -F_y \cdot y \cdot \cos(\omega_{\rm WF} t) \quad = -\frac{F_y}{\sqrt{2m\gamma\omega_y}}(a^\dagger + a)\cos(\omega_{\rm WF} t)$$

Wave function discontinuity equation

$$i\{\Psi(+;n) - \Psi(-;n)\} = V(nT)\,\Delta t\,\Psi(-;n)$$

Stroboscopic master equation

$$\Psi(-;n) = \left\{1 + i\frac{F_y \Delta t}{\sqrt{2m\gamma\omega_y}}\cos(n\omega_{\rm WF}T)\left(a^\dagger e^{-i\omega_y T} + a e^{i\omega_y T}\right)\right\}\Psi(-;n-1)e^{-i\omega_{\rm in}T}$$

Quantum mechanics of coherent oscillations cont'd

Solution

$$|\Psi(+;n)\rangle = \left\{1 + i\frac{F_y \Delta t}{\sqrt{2\gamma m \omega_y}} \left(w(n)a^\dagger + w^*(n)a\right)\right\} |\mathrm{in}\rangle \,,$$

Deja vue from classical mechanics

$$w(n) = \sum_{k=1}^{n} \cos(k\omega_{\mathrm{WF}}T) \exp\{-i(n-k)\omega_y T\}$$

$$= \frac{1}{2} \cdot \left[\frac{\exp(-in\omega_y T) - \exp(-in\omega_{\mathrm{WF}}T)}{\exp(-i(\omega_y - \omega_{\mathrm{WF}})T) - 1} \quad + \{\omega_{\mathrm{WF}} \to -\omega_{\mathrm{WF}}\}\right]$$

Quantum oscillations

Particle displacement as a quantum-mechanical expectation value

$$y(n) = \frac{1}{\sqrt{2\gamma m\omega_y}} \left\langle \Psi^*(+;n) \left| (a^\dagger + a) \right| \Psi(+;n) \right\rangle$$

$$= -i\frac{F_y \Delta t}{2\gamma m\omega_y} \left(w^*(n) - w(n)\right) \left\langle \mathrm{in} \left| [a, a^\dagger] \right| \mathrm{in} \right\rangle$$

$$= -i\frac{F_y \Delta t}{2\gamma m\omega_y} \left(w^*(n) - w(n)\right)$$

Independent of the initial quantum state of the particle because of

$$[a, a^\dagger] = 1$$

Exact replica of the classical mechanics result (Ehrenfest theorem) .

Summary

- Coherent oscillation amplitude is independent of the idle betatron motion of individual particles be it either classical or quantum
- Exemplary case of the Ehrenfest theorem: identical functional form of the RF driven amplitude of coherent oscillations of the bunch from the classical mechanics domain down to the deep quantum domain.
- Neither intrabeam scattering nor scattering off the residual gas do affect coherent oscillations
- Heisenberg uncertainty relation does not preclude observation of picometer coherent oscillation amplitudes --- the sole issue is to develope pickups capable of detection of a very weak periodic collective signal of 10^9 particles in a noisy environment
- Take advantage of 1 year observation time: accuracy propto the inverse square root of the observation time

JEDI, Phys. Rev. ST Accel. Beams 17, 052803 (2014)

520 nm (96 s) → 1.6 nm (10^7 s)

5 pm sensitivity target: a task of 300-fold improving of the sensitivity of the beam position monitors

There are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy.