CORRECTIONS OF ORDER $\alpha^2(Z\alpha)^5$ TO HYPERFINE SPLITTING AND LAMB SHIFT IN HYDROGEN AND MUONIUM

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Experiments on measurements of the ground state hyperfine splitting in muonium have reached very high accuracy [1]

$$\Delta\nu_{HFS} = 4\ 463\ 302.88\ (16)\ \text{kHz}. \tag{1}$$

Relative error of this measurement is equal to $3.6 \cdot 10^{-8}$ and a new experiment is currently in progress [2] with the goal to reduce its magnitude further to less than 10^{-8}.

Recent experiments on measurement of the $2S_{1/2} - 2P_{1/2}$ splitting in hydrogen [3, 4] also produced very accurate results

$$\Delta\nu_{2S} = 1057\ 845\ (9)\ \ \text{kHz}\,,$$

$$\Delta\nu_{2S} = 1057\ 851.4\ (1.9)\ \ \text{kHz}. \tag{2}$$

Record accuracy was achieved in the measurement ot the $1S$ Lamb shift [5]

$$\Delta\nu_{1S} = 8\ 172.86\ (6)\ \ \text{MHz}. \tag{3}$$

These spectacular experimental achievements constitute a serious challenge to the theory. Several years ago we have started a program of calculating of all experimentally significant contributions to hyperfine splitting and Lamb shift intervals with the goal of matching this challenge of experiment.

Our theoretical work on high order corrections to hyperfine splitting and Lamb shift concentrated during this period on calculation of nonrecoil contributions of order $\alpha^2(Z\alpha)^5$. Their magnitude may run up to several kilohertz for HFS in the ground state of muonium and for $n = 2$ Lamb shift in hydrogen and may be as large as tenths of kilohertz for the

ground state Lamb shift in hydrogen. Corrections of order $\alpha^2(Z\alpha)^5$ are really the only still unknown corrections to HFS which may be so large (see, e.g. [6], [7]), while for the Lamb shift in hydrogen there are some other sources of contributions of comparable magnitude (see, e.g. [8]).

We have shown that there exist six gauge invariant sets of diagrams, which produce corrections of order $\alpha^2(Z\alpha)^5$ to muonium HFS [9]. All these diagrams may be obtained with the help of different radiative insertions from the skeleton diagram, which contains two external photons attached to the electron line.

Actual calculation of matrix elements is impeded by the ultraviolet (UV) and infrared (IR) divergencies. To get rid of UV problems we work only with renormalized (subtracted) graphs and subgraphs. IR problems are as usual more difficult to deal with than the UV ones. We choose the Fried-Yennie (FY) gauge for the radiative photons. The advantage of this choice of gauge is connected with maximally mild behavior of all matrix elements in the IR region of integration momenta in the FY gauge and with the possibility to perform on-mass-shell renormalization without introduction of the infrared photon mass (see,e.g. [6]). This last feature of the FY gauge is especially important for us as we want to push analytical calculations as far forward as possible and working without the photon mass gives us the chance to obtain much simpler analytical formulae. On the other hand working in the FY gauge without the photon mass makes it absolutely necessary to pay special attention to the infrared behavior of the integrand functions and to perform cancellation of spurious IR divergences prior to integration.

Each contribution of order $\alpha^2(Z\alpha)^5$ arises from radiative insertions in the skeleton graph with two external photons attached to the electron line. Contribution to hyperfine splitting, produced by diagrams is given by the expression (see, e.g. [6])

$$\Delta E_{HFS} = 8\frac{Z\alpha}{\pi n^3}(\frac{\alpha}{\pi})^2 E_F \int_0^\infty \frac{d|\mathbf{k}|}{\mathbf{k}^2} F(\mathbf{k}), \tag{4}$$

where $k_\mu = (0, \mathbf{k})$ is the momentum of the external photons measured in the electron mass units and

$$E_F = \frac{8}{3}(Z\alpha)^4 \frac{{m_r}^3}{mM}(1 + a_\mu), \tag{5}$$

where $m_r = m/(1 + m/M)$ is the reduced mass of the electron-muon system and a_μ is the anomaly of the muon magnetic moment. Function $F(\mathbf{k})$ is connected with the numerator structure of each particular graph and describes radiative corrections to the skeleton diagram. It is normalized on the skeleton numerator contribution.

Five of six sets of diagrams relevant for the corrections of order $\alpha^2(Z\alpha)^5 m$ to HFS contain graphs with closed electron loops. Contributions induced by polarization operator insertions in external photons (a, b) and by simultaneous insertion of radiative photon in electron line and one-loop polarization operator in external photon (c) had been calculated in analytic form [9]. Correction induced by polarization operator insertions in radiative photons (d) was obtained in semianalytic form as one-dimensional integral, where integrand is itself a complete elliptic integral [10]. Contribution induced by the fifth gauge invariant set of graphs containing light by light scattering insertions (e) was reduced to a three-dimensional integral over Feynman parameters which was calculated numerically [11]. All respective results are presented in the Table 1.

Table 1

Corrections to HFS induced by the diagrams with closed electron loops

ΔE	$\times\frac{\alpha^2(Z\alpha)}{\pi n^3}E_F$	kHz (n=1)
a	$\frac{36}{35}$	0.57
b	1.87	1.03
c	−0.67	−0.37
d	−0.31	−0.17
e	−0.47	−0.26
Total	1.44	0.797

Contributions to the Lamb shift of order $\alpha^2(Z\alpha)^5$ are induced by the same six gauge invariant sets of diagrams which were just described for hyperfine structure. The only difference is that the tensor structure which is relevant for the Lamb shift differs from the one which was relevant for HFS.

General expression for the contributions to the Lamb shift of this order has the form (see, e.g. [8])

$$\Delta E_L = -\frac{16}{n^3}\, m \left(\frac{m_r}{m}\right)^3 \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\alpha)^5}{\pi} \int_0^\infty \frac{d|\mathbf{k}|}{\mathbf{k}^4}\, L(\mathbf{k})\,. \tag{6}$$

Function $L(\mathbf{k})$ as in the case of the contribution to HFS is calculated for each particular graph and describes radiative corrections to the skeleton diagram. The integral for the Lamb shift interval in eq.(6) is more singular in the infrared region than the one for the HFS in eq.(4). This singularity indicates that some diagrams contain not only contributions of order $\alpha^2(Z\alpha)^5 m$ to the Lamb shift but also contain abundantly contributions of lower order in parameter $Z\alpha$. These contributions are connected with integration over external photon momenta of characteristic atomic order $mZ\alpha$ and they should be subtracted to make calculation of corrections of order $\alpha^2(Z\alpha)^5 m$ possible [8].

Numerical calculation of contributions induced by diagrams with polarization operator insertions in the external photons and in the radiative photons as well as by the graphs containing light by light scattering insertions was performed recently [8], [12] [13] and [14] and respective results are presented in the Table 2.

Table 2

Corrections to the Lamb shift induced by the diagrams with closed electron loops

ΔE	$\times\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m$	kHz (2S)	kHz (1S)
a	−0.061	−0.33	−2.6
b	0.508	2.75	22.0
c	0.611	3.31	26.5
d	−0.073	−0.39	−3.2
e	−0.122(2)	−0.660	−5.28
Total	0.863(2)	4.67(1)	37.3(1)

Work on calculation of contributions to HFS and to the Lamb shift of order $\alpha^2(Z\alpha)^5$ which are induced by the last and most bulky set of diagrams with two radiative photons inserted in the electron line is in progress now. This set of diagrams with insertions of two radiative photons in the electron line which is relevant for calculation of corrections of order $\alpha^2(Z\alpha)^5$ to HFS and the Lamb shift contains nineteen topologically different diagrams. The simplest way to describe these graphs is to realize that they were obtained from three graphs for the two-loop electron self-energy by insertion of two external photons in all possible ways.

It should be mentioned that some diagrams contain also contributions of previous order in $Z\alpha$. Physical nature of these contributions is especially transparent in the case of HFS. They correspond to anomalous magnetic moment, their true order in $Z\alpha$ is lower than their apparent order and they should be subtracted from the matrix elements prior to calculation of the contributions to HFS. Analogous situation holds also in the case of the Lamb shift. The only difference is that this time not only the slope of the Pauli formfactor but also the slope of the Dirac formfactor of the electron is capable to produce lower order contribution to the splitting of the energy levels (see, e. g. [8], [12] and [13]) and all respective terms have to be subtracted. Technically cases of lower order contributions both to HFS and to the Lamb shift are quite similar. Lower order terms are each time proportional to the exchanged momentum squared and to get rid of them one has to subtract all low frequency terms proportional to the exchanged momentum squared in the asymptotes of the matrix elements when they exist.

Contributions both to HFS and Lamb shift induced by nine diagrams $a - h$ and l have been obtained in [15, 16] and are presented in the Table 3.

Table 3

Corrections to HFS and Lamb shift induced by the diagrams with two radiative photons

ΔE	HFS $\times\frac{\alpha^2(Z\alpha)}{\pi n^3}E_F$	Lamb $\times\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m$
a	9/4	0
b	−6.660...	2.955...
c	3.932...	−2.223...
d	−3.903...	−5.235...
e	4.566...	5.056...
f	−3.401...	−1.017...
g	2.682...	−0.146...
h	33/16	153/80
l	−1.984...	1.749...

Calculations of contributions induced by the remaining ten graphs are in the finishing stage now.

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