ANALYTIC SOLUTION OF THE DOUBLE-LOGARITHMIC INTEGRAL EQUATION WITH QCD RUNNING COUPLING

S. I. Manayenkov

The behaviour of the non-singlet quark structure function $f_{NS}(x,Q^2)$ in deep inelastic scattering (DIS) at small x is important for the description of the quark distributions in hadrons. The most popular way to predict the behaviour of $f_{NS}(x,Q^2)$ for $x \ll 1$ in the framework of the perturbative QCD consists in extrapolating into this region the Altarelli-Parisi equations which describe DGLAP evolution of the structure functions with respect to $\ln(Q^2)$ [1–3]. But these equations are valid for $x \sim 1$ where they sum up all the leading contributions of the Feynman graphs of the ladder type [3]. To sum correctly up all the leading double-logarithmic (DL) contributions in QED at $x \ll 1$ one can apply the integral equation proposed in [4] (see also review [5]). The generalization of the DL integral equation for QCD when running coupling α_s depends on momenta of virtual particles in the ladder graphs has been done in [6] where properties of numerical solutions have been discussed.

To take into account all the DL contributions to the non-singlet quark structure function $f_{NS}(x,Q^2)$ one has to solve the integral equation [6]

$$A(t,\beta) = \frac{C_F}{2\pi}\alpha_s(t/\beta) + \frac{C_F}{2\pi}\int_\beta^1 \frac{d\beta'}{\beta'}\int_{\mu^2}^{t_m}\frac{dt'}{t'}A(t',\beta')\alpha_s(t_m) \tag{1}$$

and calculate $f_{NS}(x,Q^2)$ making use of the formula

$$f_{NS}(x,Q^2) = \delta(1-x) + \int_{\mu^2}^{S_m} A(t,\tilde{\beta})\frac{dt}{t} \tag{2}$$

with $\tilde{\beta} = x(1+t/Q^2)$. Here $C_F = (N^2-1)/2N$ for the colour group $SU(N)$, μ^2 denotes the infrared cut-off parameter, $t_m = t\beta'/\beta$, $S_m = Q^2/(4x)$, $q^2 = -Q^2 < 0$, q denotes the momentum transferred in lepton-hadron DIS, $x = Q^2/(2pq)$, p is a momentum of a struck quark in a hadron. Applying the well known formula for QCD running coupling $\alpha_s(k^2) = 12\pi/[(33-2n_f)\ln(k^2/\Lambda^2)]$ and the variables $y = \ln(t/\Lambda^2)$, $y' = \ln(t'/\Lambda^2)$, $z = \ln(1/\beta)$, $z' = \ln(1/\beta')$, $b = \ln(\mu^2/\Lambda^2)$ we can rewrite equation (1) in the form

$$A(y,z) = \frac{g}{y+z} + \int_0^z dz' \frac{g}{y+z-z'}\int_b^{y+z-z'} dy' A(y',z') \tag{3}$$

where $g = 8/(33-2n_f)$. It is easy to check that the DL integral equation (3) is equivalent to the differential equation

$$y\frac{\partial^2 A}{\partial y^2} - y\frac{\partial^2 A}{\partial y \partial z} + \frac{\partial A}{\partial y} - \frac{\partial A}{\partial z} + gA = 0 \tag{4}$$

with boundary conditions valid when $b \neq 0$

$$A(y,0) = \frac{g}{y}\,, \tag{5}$$

$$\left[\frac{\partial A(y,z)}{\partial y} - \frac{\partial A(y,z)}{\partial z}\right]\Big|_{y=b} = 0\,. \tag{6}$$

Applying the Laplace transformation $\phi(y,\rho) = \int_0^\infty e^{-\rho z} B(y,z)dz$ to equation (4) and remembering the boundary condition (5) we get

$$y\frac{\partial^2\phi}{\partial y^2} + (1-\rho y)\frac{\partial\phi}{\partial y} - (\rho - g)\phi = 0\,. \tag{7}$$

The general solution of (7) looks like ($\lambda = g/\rho$)

$$\phi(y,\rho) = (\rho y)^{-1/2}\exp\{\rho y/2\}[C(\rho)W_{\lambda-1/2,\,0}(\rho y) + S(\rho)W_{1/2-\lambda,\,0}(-\rho y)] \tag{8}$$

where $W_{k,\,m}(x)$ denotes the Whittaker's function [7]. It is easy to get the unequality $|A| \le (g/b)\exp\{gz\}$ from (3) which lead to the relation $|\phi| \le (g/b)/(\rho - g)$. The latter unequality is compatible with (8) if $S(\rho) \equiv 0$ as $W_{1/2-\lambda,\,0}(-\rho y)$ grows at large ρy exponentially in accordance with the asymptotical relation [7]

$$W_{k,\,m}(\pm x) = (\pm x)^k e^{\mp x/2}[1 + O(1/x)]\,. \tag{9}$$

After that the only unknown function $C(\rho)$ can be found from the boundary condition (6) and is equal to $g\exp\{-\rho b/2\}/W_{\lambda,\,1/2}(\rho b)$. Finally the general solution of (3) at $b \neq 0$ is given by the inverse Laplace transformation

$$A(y,z) = \frac{g}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\rho(\rho y)^{-1/2}\exp\{\rho(z + y/2 - b/2)\}W_{\lambda-1/2,\,0}(\rho y)/W_{\lambda,\,1/2}(\rho b)\,. \tag{10}$$

The arbitrary parameter a in (10) should be chosen so that for $Re\ \rho \ge a$ the integrand has no singularities.

It is easy to check that for small z ($\beta \sim 1$) when large ρ contribute to integral (10) and the asymptotical formula (9) for $W_{k,m}(x)$ is applicable we get the well known result

$$A(y,z) = \frac{g}{y}I_0(2\sqrt{gz\ln(y/b)}) \tag{11}$$

where $I_n(x)$ denotes the Infeld function [7]. Formula (11) corresponds to the DGLAP evolution.

For $\mu = \lambda$ ($b = 0$) we have the relation $1/W_{\lambda,\,1/2}(0) = \Gamma(1-\lambda)$. Substituting this relation into (10) and using of the Cauchy theorem for the calculation of the integral in (10) we get another special solution

$$A(y,z) = \frac{g}{y+z} + g^2\int_\theta^\infty \frac{s\,ds}{e^s - 1}\sum_{n=0}^\infty \frac{g^n z^n}{n!(n+1)!}\left[s - \frac{y}{z}\frac{s}{e^s - 1}\right]^n\,. \tag{12}$$

where $\theta = \ln(1 + y/z)$. It is easy to prove that at $z \gg 1$ and $y \le z$ the main contribution to integral (12) gives the term

$$A_0(y,z) = g^2\int_0^\infty e^{-s}\sum_{n=0}^\infty \frac{g^n z^n s^{n+1}}{n!(n+1)!}ds = g^2 e^{gz}\,, \tag{13}$$

and corrections to this simple expression are much smaller than A_0 (corrections are $\sim \exp\{gz/2\}$ and $\sim \exp\{2\sqrt{gy}\}$). The calculation of $f_{NS}(x,Q^2)$ by using (2) and (13) predicts at $x \ll 1$, $Q^2/\mu^2 \gg 1$ the following behaviour

$$f_{NS}(x,Q^2) = g^2 x^{-g}[\ln(Q^2/\mu^2) - \psi(g) + \psi(1)] \tag{14}$$

where $\psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$ denotes the logarithmic derivative of gamma-function [7] and $g = 8/(33 - 2n_f) = 0.32$ for four active flavours ($n_f = 4$) and $g = 0.296$ for $n_f = 3$. Result (14) is in a contradiction with the behaviour $f_{NS} \sim x^{-\gamma}$ with $\gamma = \sqrt{g \ln(y/b)/\ln(1/x)}$ predicted for the solution (11) (Altarelli-Parisi evolution). Formula (14) is in an agreement with the numerical calculation of $f_{NS}(x, Q^2)$ performed in [6].

I am grateful to Prof. M.G.Ryskin for useful discussions and a constant interest to the present work.

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THE ROLE OF SCREENING CORRECTIONS IN SMALL x-BEHAVIOUR OF STRUCTURE FUNCTIONS

M. G. Ryskin, Yu. M. Shabelski

We consider the influence of shadowing effects on the proton structure function in the small-x interval. The gluon-gluon shadowing are noticeable in the HERA kinematical region while the screening of the quark component of the structure function effects negligibly the gluon distribution. The only noticeable effect is the decreasing of sea quark densities at small-x. The explicit form of the gluon distribution in the proton depends significantly on the form of used boundary condition at $Q^2 = Q_0^2$. We consider also difference of results obtained with the help of Kuraev-Lipatov-Fadin and Altarelli-Parisi evolution equations.

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